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SPACES WITH ZERO SET BASES

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Abstract: Answering a question raised by M. Katětov, we construct a regular space that is not completely regular, yet has a basis consisting of interiors of zero sets.

Key words: Regular space, completely regular space, zero set.

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In this note we answer a question of M. Katětov (see [K], p. 105, Remark 5.3) by constructing a regular non-completely regular topological space that has a basis consisting of interiors of zero sets. Define the FR-index of a topological space to be the smallest cardinal, $\varkappa$, such that $\{U$: U is the interior of the intersection of not more than $\varkappa$ zero sets$\}$ forms a base for the space (see [K]). Katětov originally asked whether for each $\varkappa$ there exists a regular, non-completely regular space with FR-index $\varkappa$ and gave an affirmative answer in the cases $\varkappa \geq \omega_1$. Our space gives an answer to the remaining case of $\varkappa = 1$.

The space Y . We will first construct a regular non-normal space, Y, and then show how to use Y to construct the

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desired example, Z. Let R^o and R^1 be two disjoint copies of the real line. The space Y consists of all points in the open upper half plane, $\{(x,y): x,y \in R, 0< y\}$, together with all points of $R^o \cup R^1$. Each point of $Y \setminus (R^o \cup R^1)$ is declared isolated. If $x \in R^o$, and n is a non-negative integer, we define B(x,n), the n^{th} basic open neighborhood of x, to be $\{x\} \cup S$ where S is the open line segment in the upper half plane that has lower endpoint x, is 1/n units long, and makes an angle of $\pi/4$ with the x-axis. If $x \in R^1$, then B(x,n) = $= \{x\} \cup S$ where S is as above (i.e. S is the open line segment in the upper half plane that has lower endpoint x and is 1/n units long) except that S makes an angle of $3\pi/4$ with the x-axis. Wit the topology generated by the above base, Y is a completely regular space, but is not normal since the two closed sets R^o and R^1 can't be separated.

Notation. The non-negative integers are denoted by ω. Let F be the set of all functions, f, such that

i) for some $n \in \omega$, f is a function from $\{n, n+1, \dots\}$ into ω, and

ii) $f(k) = 0$ for all but finitely many $k \in \omega$.

Next we need a convenient notation for certain subsets of $Y \times F$ and F. For each $f \in F$, if $A \subset Y$ let A_f denote $A \times \{f\} \subset Y \times F$ and if $y \in Y$ let y_f denote the ordered pair $(y,f) \in Y \times F$. For $f \in F$, put $f^* = \{g \in F: g|_{\mathrm{dom}\, f} = f$ and $|\mathrm{dom}\, g \setminus \mathrm{dom}\, f| = 1$. (Here dom f denotes the domain of the function f and $g|_{\mathrm{dom}\, f}$ is the restriction of the function g to dom f .) For each $f \in F$ we will often be concerned with the minimum value of dom f which we denote by μf. The symbol πf is used for

μf mod 2, i.e. πf is 0 if μf is even and 1 if μf is odd.

The space Z. Let ∞ be a point that is not in $Y \times F$ and set $Z = \{\infty\} \cup Y \times F$. Using the basic open sets, $B(y,n)$ of Y and the notation given above we define a basis for Z. For each $n \in \omega$ and $f \in F$ declare the following to be basic open subsets of Z:

1) $\{y_f\}$ where $y \in Y \setminus R^{\pi f}$,

2) $B_f(x,n) \cup \bigcup \{B_g(x,n) : g \in f^* \text{ and } g(\mu g) > n\}$ where $x \in R^{\pi f}$, and

3) $B(\infty,n) = \{\infty\} \cup \bigcup \{Y_g : \mu g > n\} \cup \bigcup \{Y_g \setminus R_g^{\pi g} : \mu g = n\}$.

The reader can check that the above collection of basic open sets does indeed form a basis for a Hausdorff topology on Z. The basic sets of form (1) and (2) are clopen and sets of the form (3), while not clopen, do have the property that $cl(B(\infty, n+1)) \subset B(\infty, n)$. Hence Z is regular.

Verification of the properties of Z. We first show that the basis of Z consists of interiors of zero sets. Basic open sets of the form (1) and (2) are clopen and hence interiors of zero sets. For sets of the form (3), note that the interior of $\{\infty\} \cup \bigcup \{Y_g : \mu g \geq n\}$ is just $B(\infty,n)$. The set $\{\infty\} \cup \bigcup \{Y_g : \mu g \geq n\}$ is the zero set of the continuous function $g: Z \to R$ defined by

$$g(x_f) = \begin{cases} 1/f(n-1) & \text{if } n-1 \in \text{dom } f \\ 0 & \text{otherwise.} \end{cases}$$

Next we show that Z is not completely regular by proving that the point ∞ and the closed set $C = \bigcup \{Y_f :$

: dom $f = \omega\}$ cannot be separated by a continuous function. Indeed, suppose $h: Z \longrightarrow R$ is continuous and $h(\infty) = 0$. We must show $h(C) \neq 1$. This follows easily once it is realized that if h is small on a large part of Y_f, then it is small on a large part of $Y_{f'}$, for some $f' \in f^*$. More precisely:

<u>Assertion</u>. Assume $f \in F$ with dom $f \neq \omega$, U is a nonempty interval in $R^{\pi f}$, and $h(U_f) \leq b$. Then for each $\varepsilon > 0$, there exists $f' \in f^*$ and U', a nonempty subinterval of $R^{\pi f}$ such that $h(U'_{f'}) \leq b + \varepsilon$.

<u>Proof</u>. Fix f, U and b as above and let $\varepsilon > 0$. Since h is continuous, for each $x \in U$ there exists an n_x such that $h(B_f(x,n_x)) \leq b + \varepsilon$. By the Baire category theorem, there is an m such that $\{x \in U \mid n_x = m\}$ is not nowhere dense (in the usual order topology on $R^{\pi f}$). Define $f' \in f^*$ by

$$f'(k) = \begin{cases} f(k) & \text{if } k \in \text{dom } f \\ m & \text{if } k = \mu f - 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then the above conditions and the definition of the topology of Z imply that $cl(h^{-1}((-\infty, b + \varepsilon]))$ contains a nonempty interval, U', of $R_{f'}^{\pi f'}$. Hence $h(U'_{f'}) \leq b + \varepsilon$. □

Now we can show $h(C) \neq 1$. Since $h(\infty) = 0$, there is an n such that $h(B(\infty, n)) < 1/4$ and hence an $f \in F$ with $h(Y_f) < 1/4$. If dom $f \neq \omega$, apply the assertion repeatedly until an f' and U' are obtained with dom $f' = \omega$ and $h(U'_{f'}) < 1/2$. But dom $f' = \omega$ implies $U'_{f'} \subset C$, so $h(C) \neq 1$.

<u>Remarks</u>. Note that we could have tried performing our

procedure on any non-normal space instead of Y (using two disjoint closed sets that can't be separated in place of R^0 and R^1). However, with many non-normal spaces we would not obtain the desired counterexample. If the Tychonoff plank, for example, is run through our procedure, the resulting space is completely regular. Petr Simon has pointed out that any completely regular non-normal space can be used to give the desired counterexample if it is changed slightly before running it through the above procedure. Here is his clever modification: Let X be any completely regular non-normal space and let H and K be two disjoint closed subsets of X that cannot be separated by disjoint open sets. Let X^1, X^2, X^3, and X^4 be four disjoint copies of X. Let Y be the space obtained from $X^1 \cup X^2 \cup X^3 \cup X^4$ by identifying H^1 with H^2, K^2 with K^3, H^3 with H^4 and K^4 with K^1. Now Y can be run through the our procedure using $R^0 = H^1 \cup K^2$ and $R^1 = H^3 \cup K^4$ to obtain the desired counterexample.

R e f e r e n c e

[K] M. KATĚTOV: Prostranstva, opredeljaemye zadaniem semejstva centrirovannych sistem, Uspechi matematičeskich nauk, t. XXXI, vyp. 5 (191), (1976).

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