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SEQUENTIAL REGULARIZATION OF ILL-POSED PROBLEMS INVOLVING UNBOUNDED OPERATORS

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Abstract: Let $A:D(A)\longrightarrow H$ be a closed densely defined linear operator in a real Hilbert space H and suppose that for a certain $f\in H$ the ill-posed problem $Au = f$ has a unique solution u. Let B be a bounded positive definite operator on H and set $u_0 = 0$. Then for $n = 1,2,\ldots$ the well-posed problem
$\langle Au_n, Av\rangle + \langle Bu_n, v\rangle = \langle Bu_{n-1}, v\rangle + \langle f, Av\rangle, \quad \forall v \in D(A)$
has a unique solution $u_n\in D(A)$ and $u_n \longrightarrow u$ as $n\longrightarrow\infty$.

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1. Introduction. Suppose that H is a real Hilbert space and $D(A)\subset H$ is a dense subspace. This paper is a theoretical study of a method of approximating the solution of the problem

$$Au = f \tag{1}$$

where $f\in H$ and $A:D(A)\longrightarrow H$ is a closed unbounded operator. We assume that for a certain $f\in H$ the problem (1) has a unique solution u, without assuming that A is an isomorphism of D(A) onto the range of A. It is then well-known that equation (1) is ill-posed, that is, for small perturbations of the equation

$$Ax = f + \delta f$$

may have no solution at all, or may have a solution x which is not near to the solution u of equation (1). We will show that the solution of (1) may be approximated by a <u>sequence</u> of solutions of associated well-posed problems. The idea of replacing a problem of type (1) by a family of nearby well-posed problems has been studied extensively by Lattes and Lions [3] under the title "quasi-reversibility". In particular Lattes and Lions [3, p. 289] show that the problem

$$(2) \quad \langle Au_\varepsilon, Av\rangle + \varepsilon\langle u_\varepsilon, v\rangle = \langle f, Av\rangle, \quad \forall v \in D(A)$$

is well-posed for each $\varepsilon > 0$ and the solutions u_ε of (2) converge to the solution u of (1) as $\varepsilon \to 0$. In solving (2) one must in essence "invert the operator $\varepsilon I + A^* A$", which depends on the parameter ε. In this paper we will replace equation (1) by a sequence of well-posed problems the solution of which requires the inversion of a <u>single</u> operator which is <u>independent</u> of the parameter. The method considered here is related to an analogous procedure for bounded operators studied by Kryanev [2].

As an example of a specific problem of type (1) Lattes and Lions [3, p. 290] consider the boundary value problem

$$Au = 0$$

$$u\big|_{\Gamma_0} = g_0$$

$$\frac{\partial u}{\partial \nu_A}\bigg|_{\Gamma_0} = g_1 \qquad \text{(conormal derivative)}$$

where Γ_0 is the boundary of an open domain $\Omega \subset R^n$ and A is a second order differential operator in Ω given by

$$Au = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0 u$$

where $a_{ij} \in C^3(\bar{\Omega})$, $a_0 \in C^0(\bar{\Omega})$,

$$\sum_{i,j=1}^{m} a_{ij}(x)\,\xi_i\,\xi_j \geq \alpha_1(\xi_1^2 + \ldots + \xi_n^2), \quad \alpha_1 > 0$$

and

$$\alpha_0(x) \geq \alpha_0 > 0.$$

This problem is analyzed by finding a function $\Phi \in H^2(\Omega)$ such that

$$\Phi\big|_{\Gamma_0} = g_0, \quad \frac{\partial \Phi}{\partial \nu_A}\Big|_{\Gamma_0} = g_1$$

and considering the problem satisfied by $w = u - \Phi$:

$$Aw = f$$

$$w\,\big|_{\Gamma_0} = 0$$

$$\frac{\partial w}{\partial \nu_A}\Big|_{\Gamma_0} = 0$$

where $f = -A\Phi$. The domain of the unbounded operator A is then given by

$$D(A) = \{v \in L_2(\Omega) : Av \in L_2(\Omega),\ \ v\,\big|_{\Gamma_0} = 0,\ \frac{\partial v}{\partial \nu_A}\Big|_{\Gamma_0} = 0\}.$$

For details the reader is referred to Lattes and Lions [3].

2. <u>The regularization Procedure</u>. Kryanev [2] investigated the iterative procedure

$$Bx_n + Ax_n = Bx_{n-1} + f$$

for approximating solutions to the ill-posed problem

$$Ax = f$$

where A is a bounded positive semi-definite linear operator on a Hilbert space H and B is a bounded positive definite operator on H which is chosen to improve the conditioning of the operator B + A. However, as noted above, many ill-posed problems which are of practical interest may be formulated as an equation of type (1) where A is a closed, densely defined but unbounded operator on a suitable Hilbert space. We will examine Kryanev's procedure in the context considered by Lattes and Lions. Below, $A:D(A) \longrightarrow H$ will be a closed linear operator defined on the dense subspace D(A) of the real Hilbert space H and B will be a bounded linear operator on H satisfying

$$\langle Bx,x \rangle \geq c \|x\|^2, \quad c > 0.$$

We recall that the domain $D(A^*)$ of the adjoint operator is by definition the set of all vectors $y \in H$ for which there is a $y^* \in H$ satisfying

$$\langle Ax,y \rangle = \langle x,y^* \rangle , \quad \forall\, x \in D(A)$$

and the adjoint operator A^* is defined by $A^* y = y^*$.

First we state a lemma which will be useful in the sequel.

Lemma 1. The operator $B + A^* A$ has a bounded inverse $U = (B + A^* A)^{-1}:H \longrightarrow D(A^* A)$ which is positive.

Proof. By assumption there is a number $c > 0$ such that

$\langle Bx,x\rangle \geq c\|x\|^2$ for each $x \in H$. Choose $k>0$ such that $\max\{|kc-1|, k\|B\|\} < 1$. Let $\bar{A} = kA$, then by a theorem in Riesz and Sz.-Nagy [5, p. 307], $(I + \bar{A}^*\bar{A})^{-1}: H \longrightarrow D(A^*A)$ exists and $\|(I + \bar{A}^*\bar{A})^{-1}\| \leq 1$. Now,

$\|(kB - \bar{A}^*\bar{A}) - (I + \bar{A}^*\bar{A})\| \leq \max\{|kc-1|, k\|B\|\} < 1$,

and it follows by a standard perturbation result (see e.g. [4, p. 45]) that

$$kB + \bar{A}^*\bar{A} = k(B + A^*A)$$

is invertible. Hence $B + A^*A$ is invertible and it can be shown that $\langle (B + A^*A)^{-1}x,x\rangle \geq 0$ for all $x \in H$ as in [5, p. 308].

The next lemma defines a sequence of well-posed problems the solutions of which we shall show converge to the solution u of equation (1).

Lemma 2. Set $u_0 = 0$, then for $n = 1,2,\ldots$, the problem

(3) $\langle Bu_n, v\rangle + \langle Au_n, Av\rangle = \langle Bu_{n-1}, v\rangle + \langle f, Av\rangle, \ \forall v \in D(A)$

has a unique solution $u_n \in D(A)$ which depends continuously on f.

Proof. Since A is a closed linear operator, the subspace $D(A)$ endowed with the norm

$$\|x\|_{D(A)} = (\|x\|^2 + \|Ax\|^2)^{1/2}$$

and corresponding inner product

$$\langle x,y\rangle_{D(A)} = \langle x,y\rangle + \langle Ax,Ay\rangle$$

is a Hilbert space. Define the symmetric bilinear form $Q(x,y)$ on $D(A)\times D(A)$ by

$$Q(x,y) = \langle Bx,y\rangle + \langle Ax,Ay\rangle .$$

It is easy to see that $Q(x,y)$ is continuous (with respect to the norm $\|\cdot\|_{D(A)}$) and for $x \in D(A)$

$$Q(x,x) \geq \min(c,1) \|x\|^2_{D(A)}.$$

Hence $Q(x,y)$ is coercive and the existence of u_n follows by use of the Lax-Milgram lemma (see e.g. [1, p.41]). Furthermore, if

$$Q(u_n,v) = \langle Bu_{n-1},v\rangle + \langle f,Av\rangle, \quad \forall v \in D(A)$$

and

$$Q(u_n',v) = \langle Bu_{n-1},v\rangle + \langle f',Av\rangle, \quad \forall v \in D(A),$$

then setting $v = u_n - u_n'$, we obtain

$$\begin{aligned}\min(c,1)\|u_n - u_n'\|^2_{D(A)} &\leq Q(u_n - u_n', u_n - u_n') \\ &= \langle f - f', A(u_n - u_n')\rangle \\ &\leq \|f - f'\| \; \|u_n - u_n'\|_{D(A)} .\end{aligned}$$

From this it follows that u_n is unique and the mapping $f \longmapsto u_n$ is continuous.

The main result may now be stated.

Theorem. The solutions u_n of the well-posed problems (3) converge strongly to the solution u of problem (1).

Before proceeding with the proof we note that

$$u_n = UBu_{n-1} + u_1 \tag{4}$$

where $U = (B + A^* A)^{-1}$. In fact, we have by Lemma 2

$$\langle A(UBu_{n-1}) + u_1),Av\rangle + \langle B(UBu_{n-1} + u_1), v\rangle$$

$$= \langle AUBu_{n-1},Av\rangle + \langle BUBu_{n-1},v\rangle + \langle f,Av\rangle =$$

$$= \langle (A^* A + B)UBu_{n-1}, v \rangle + \langle f, Av \rangle$$
$$= \langle Bu_{n-1}, v \rangle + \langle f, Av \rangle, \qquad \forall v \in D(A).$$

Equation (4) now follows by the uniqueness statement in Lemma 2. The proof of the theorem requires two further lemmas.

<u>Lemma 3</u>. For $m = 1,2,\ldots, \langle Bu_m, u_m - u \rangle \leq 0$.

Proof. Note that by Lemma 2 and equation (1), we have for all $v \in D(A)$

$$\langle Bu_m, v \rangle + \langle A \sum_{n=1}^{m} (u_n - u), Av \rangle$$
$$= \sum_{n=1}^{m} \{ \langle B(u_n - u_{n-1}), v \rangle + \langle A(u_n - u), Av \rangle \}$$
$$= 0 \text{ , for } m = 1,2,\ldots \text{ .}$$

Hence it suffices to show that

$$\text{(5)} \qquad \langle A \sum_{n=1}^{m} (u_n - u), A(u_m - u) \rangle \geq 0, \quad m = 1,2,\ldots \text{ .}$$

Note that

$$\langle A(u - UBu), Av \rangle + \langle B(U - UBu), v \rangle$$
$$= \langle f, Av \rangle - \langle (A^* A + B)UBu - Bu, v \rangle$$
$$= \langle f, Av \rangle, \qquad \forall v \in D(A)$$

and it follows from Lemma 2 that

$$\text{(6)} \qquad u_1 = u - Wu$$

where $W = UB$. We therefore have by (4) and (6)

$$W(u_{m-1} - u) = Wu_{m-1} + u_1 - u$$
$$= u_m - u,$$

and hence for $j \leq m$ we have

$$u_m - u = W^{m-j}(u_j - u).$$

Therefore, for $j < m$,

$$\langle A(u_m - u), A(u_j - u)\rangle = \langle AW^{m-j}(u_j - u), A(u_j - u)\rangle$$
$$= \langle A^* AW^{m-j}(u_j - u), u_j - u\rangle .$$

But $\langle A^* AW^k x, x\rangle = \langle A^* AW^k x, (I + B^{-1}A^* A)^k W^k x\rangle$

$$= \langle A^* AW^k x, W^k x\rangle + \sum_{j=1}^{k} \binom{k}{j} \langle A^* AW^k x, (B^{-1}A^* A)^{j-1}B^{-1}A^* AW^k x\rangle ,$$

and it is easy to show that $(B^{-1}A^* A)^n B^{-1}$ is positive for $n = 0,1,2,\ldots$, and hence $\langle A(u_m - u), A(u_j - u)\rangle \geq 0$, which proves the lemma.

From the above lemma it follows that the sequence $\{u_n\}$ is bounded, indeed

(7) $\quad e \|u_n\|^2 \leq \langle Bu_n, u_n\rangle \leq \langle Bu_n, u\rangle \leq \|B\| \; \|u_n\| \; \|u\| .$

<u>Lemma 4</u>. As $n \to \infty$, $Au_n \to Au$.

Proof. Setting $v = u_n - u$ in the equation

$$\langle A(u_n - u), Av\rangle = \langle B(u_{n-1} - u_n), v\rangle$$

and summing we obtain

(8) $\quad \sum_{n=1}^{m} \| A(u_n - u)\|^2 = \langle Bu_m, u\rangle - \sum_{n=1}^{m} \langle B(u_n - u_{n-1}), u_n\rangle .$

If we define a new inner product and norm by

$$(x,y) = \langle Bx, y\rangle \quad \text{and} \quad \|x\|_B^2 = (x,x),$$

then

$$\sum_{n=1}^{m} \langle B(u_n - u_{n-1}), u_n\rangle = \sum_{n=1}^{m} \{ \|u_n\|_B^2 - (u_{n-1}, u_n)\}$$
$$= \frac{\|u_1\|_B^2}{2} + \frac{\|u_m\|^2}{2} +$$

$$\frac{1}{2}\sum_{n=2}^{m}\{\|u_n\|_B^2 - 2(u_{n-1},u_n) + \|u_{n-1}\|_B^2\}.$$

Therefore

$$\sum_{n=1}^{m}\langle B(u_n - u_{n-1}),u_n\rangle \geq 0$$

and it follows from (8) and (7) that

$$\sum_{n=1}^{m}\|A(u_n - u)\|^2 \leq \langle Bu_m,u\rangle \leq \|B\|^2\|u\|^2/c,$$

which proves the lemma.

Finally we are in a position to complete the proof of the theorem. Since any subsequence of $\{u_n\}$ is bounded, we can extract a subsequence which converges weakly to an element $z \in H$. Since the graph of A is closed and convex, it is weakly closed and therefore $z \in D(A)$ and from Lemma 4 we have $Az = Au = f$. But the solution to problem (1) is unique, therefore $z = u$. Hence we see that any subsequence of $\{u_n\}$ in turn contains a subsequence which converges weakly to u. It follows that the entire sequence $\{u_n\}$ converges weakly to u. By Lemma 3 we have $\langle Bu_m,u_m - u\rangle \leq 0$ and hence

$$\begin{aligned} c\|u_m - u\|^2 &\leq \langle B(u_m - u),u_m - u\rangle \\ &= \langle Bu_m,u_m - u\rangle - \langle Bu,u_m - u\rangle \\ &\leq -\langle Bu,u_m - u\rangle \longrightarrow 0. \end{aligned}$$

Therefore, $u_m \longrightarrow u$, completing the proof of the theorem.

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