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ON NATURAL MEROTOPIES

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Abstract: A natural merotopy is defined and the conditions under which the merotopy is natural are found and discussed. An example of a metric space whose natural merotopy admits the value 2 for the local merotopic character is given.

Key words and phrases: Topological space, closure space, semi-separated space, merotopic space, local merotopic character, E-compact space, projective (inductive) generation.

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We shall deal with the category of merotopic spaces. This type of continuity structure has been studied under various names: quasi-uniform spaces [7], merotopic spaces [8],[9],[10], [12], quasi-nearness spaces [1],[2],[4],[5],[6]. The present paper is a free continuation of [12]. In the first part, we shall briefly summarize the definitions and basic propositions; for the details, see [9] and [12]. Then the necessary and sufficient condition for a merotopy to be natural is given. The third part contains an example to the question posed in [12], whether there exists a natural merotopy for a metric space with the value 2 for local merotopic character. Finally, the consequences of the equality $\mathcal{Mer}(X,Y) = \mathcal{C}(X,Y)$ is briefly discussed in the fourth part. The notation and symbols from [3] is used.

1. Let E be a set. If $\mathcal{A}$ and $\mathcal{B}$ are subsets of exp E, we shall say that $\mathcal{A}$ corefines $\mathcal{B}$ if for every $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ with $B \subset A$.

A merotopic space is a pair $\langle E, \Gamma \rangle$, where E is a set and $\Gamma \subset \exp \exp E$ satisfies

(i) if for $\mathcal{M} \subset \exp E$ there is some $\mathcal{N} \in \Gamma$ such that $\mathcal{N}$ corefines $\mathcal{M}$, then $\mathcal{M} \in \Gamma$;

(ii) if $\mathcal{M}_1 \cup \mathcal{M}_2 \in \Gamma$, then either $\mathcal{M}_1 \in \Gamma$ or $\mathcal{M}_2 \in \Gamma$;

(iii) for every $x \in E$, $\{\{x\}\} \in \Gamma$;

(iv) $\emptyset \notin \Gamma$, $\{\emptyset\} \in \Gamma$.

The system Γ is called a merotopy and its members are called micromeric.

A mapping $f: \langle E_1, \Gamma_1 \rangle \longrightarrow \langle E_2, \Gamma_2 \rangle$ is called a merotopic mapping if $f[\mathcal{M}_1] \in \Gamma_2$ whenever $\mathcal{M}_1 \in \Gamma_1$. The category of merotopic spaces with the morphisms just described will be denoted by $\mathbf{Mer}$, a family of all merotopic mappings from a merotopic space X to Y will be denoted by $\mathcal{M}er(X,Y)$.

Let Γ be a merotopy on a set E. A system Θ, $\Theta \subset \Gamma$ will be called fundamental (for Γ) if $\Gamma \subset \Gamma_1$ whenever Γ_1 is a merotopy on E containing Θ.

A merotopic space $\langle E, \Gamma \rangle$ will be called a filter-merotopic (and Γ a filter-merotopy) if there exists a fundamental system for Γ consisting of filters on E.

A merotopic cover (equivalently, Γ-cover) $\mathcal{Z}$ of a space $\langle E, \Gamma \rangle$ is such a cover of the set E that for each $\mathcal{M} \in \Gamma$ there exist a $Z \in \mathcal{Z}$ and an $M \in \mathcal{M}$ with $M \subset Z$.

A merotopic space $\langle E, \Gamma \rangle$ will be called semi-separated if $\{\{x,y\}\} \in \Gamma$ implies $x = y$, for each $x, y \in E$.

Let $\langle E, \Gamma \rangle$ be a merotopic space, define a mapping $cl(\Gamma)$: exp $E \longrightarrow$ exp E by the rule

$cl(\Gamma)X = \{x \in E: (\exists \mathcal{M} \in \Gamma)(\forall M \in \mathcal{M})(x \in M \ \& \ M \cap X \neq \emptyset)\}$.

It is easy to verify that $cl(\Gamma)$ is a closure operator on E, but not necessarily topology. Call it to be induced by the merotopy Γ. Obviously, if $\langle E, \Gamma \rangle$ is semi-separated, then the induced closure is semi-separated.

Denote by $\mathbb{T}\mathrm{op}_{T_1}$ ($\mathbb{C}\mathrm{l}_{T_1}$) the category of semi-separated topological (closure) spaces, and let, as usual, $\mathcal{C}(X,Y)$ be the set of all continuous mappings from X into Y.

Let $\langle E,u \rangle$ be a topological or closure space. Let mer(u)= $= \{\mathcal{M} \subset$ exp E: there is a point $x \in E$ whose neighborhood system corefines $\mathcal{M}\}$. One can check that mer(u) is a merotopy, which is filter. If u is semi-separated, then cl(mer(u)) = u. In all cases when a topological (closure) space $\langle E,u \rangle$ will be considered as a merotopic space and the merotopy will not be explicitly described, we shall assume it to be mer(u).

The category **Mer** is isomorphic to the category Q-**Near** of quasi-nearness spaces (see e.g. [5], Theorem 3.7).

2.

2.1. <u>Definition</u>. Let $\langle E,u \rangle$ be a semi-separated topological space, let Γ be a merotopy on E. We shall call a merotopy Γ to be natural, if there exists an embedding $\mathbb{F}: \mathbb{T}\mathrm{op}_{T_1} \longrightarrow \mathbb{M}\mathrm{er}$ such that

(i) $\langle E,\Gamma \rangle = \mathbb{F}\langle E,u \rangle$;

(ii) for every $\langle F,v \rangle \in \mathbb{T}\mathrm{op}_{T_1}$, if $\langle F,\Delta \rangle = \mathbb{F}\langle F,v \rangle$, then $cl(\Delta) = v$;

(iii) for every $\langle F,v\rangle$, $\langle F',v'\rangle \in \mathbb{T}\mathrm{op}_{T_1}$ and $f\colon F \to F'$, $f \in \mathcal{C}(\langle F,v\rangle, \langle F',v'\rangle)$ if and only if $f \in \mathcal{M}er\,(\mathbb{F}\langle F,v\rangle, \mathbb{F}\langle F',v'\rangle)$.

In other words, $\langle E,\Gamma\rangle$ is an image of $\langle E,u\rangle$ under some functor which is a realization of $\mathbb{T}\mathrm{op}_{T_1}$ into $\mathbb{M}\mathrm{er}$. According to the isomorphism between $\mathbb{M}\mathrm{er}$ and Q - $\mathbb{N}\mathrm{ear}$, we can similarly speak about natural quasi-nearness structures. It is well-known that topological nearness spaces are natural ([5], 4.5).

Another example of a natural quasi-nearness structure is, for a given topological space $\langle X,u\rangle$, the structure ξ defined as follows: $\mathcal{A} \in \xi$ iff there are some $A \subset X$ and $x \in uA$ such that $\mathcal{A}$ corefines $\{A, \{x\}\}$.

Let $\langle X,u\rangle$ be a topological space, let Γ be a merotopy whose fundamental system consists of all $\{F \cup \{x\} : F \in \mathcal{F}\}$ with $\mathcal{F}$ an ultrafilter on X converging to x. Γ is a natural merotopy.

Various seemingly "nice" merotopies need not be natural: On the real line $\mathbb{R}$, let $\mathcal{M} \in \Gamma$ iff there is some $x \in \mathbb{R}$ such that either the family $\{[x,x+r[: r>0\}$ or the family $\{]x-r,x] : r>0\}$ corefines $\mathcal{M}$. (The mapping x.sinx though continuous, is not merotopic.)

Let us notice the following two easy facts:

2,2. __Proposition.__ Let $\langle E,\Gamma\rangle$ be a semi-separated merotopic space, $\langle E',u'\rangle$ semi-separated topological space, let f be a mapping from E into E'. Then the following are equivalent:

(a) $f\colon \langle E, \mathrm{ess}\,\Gamma\rangle \to \langle E', \mathrm{mer}(u')\rangle$ is merotopic,

(b) $f\colon \langle E, \mathrm{cl}(\Gamma)\rangle \to \langle E', u'\rangle$ is continuous.

(ess Γ is the smallest merotopy containing $\{\mathcal{M} \in \Gamma : \bigcap \mathcal{M} \neq \emptyset\}$.)

Proof. Suppose f to be merotopic. For $X \subset E$ and $x \in cl(\Gamma)X$ let $\mathcal{M}$ be the Γ-micromeric collection with $x \in \bigcap \mathcal{M}$ and $M \cap X \neq \emptyset$ for each $M \in \mathcal{M}$. Clearly $\mathcal{M} \in ess\,\Gamma$, hence $f[\mathcal{M}] \in$ $\in mer(u')$. The collection $f[\mathcal{M}]$ witnesses to $f(x) \in cl(mer(u'))$ $f[X]$, thus f is continuous.

Suppose f to be continuous. Denote by $\mathcal{O}(x)$ the neighborhood system of x, $\mathcal{U}(f(x))$ the neighborhood system of f(x). Let $\mathcal{M} \in ess(\Gamma)$. Since $cl(ess(\Gamma)) = cl(\Gamma)$, there exists some $x \in E$ such that $\mathcal{O}(x)$ corefines $\mathcal{M}$. Since f is continuous, $\mathcal{U}(f(x))$ corefines $f[\mathcal{O}(x)]$. So $\mathcal{U}(f(x))$ corefines $f[\mathcal{M}]$ and $f[\mathcal{M}]$ belongs to $mer(u')$.

2.3. Proposition. Let $\langle E,u \rangle$ be a semi-separated non-discrete topological space, x non-isolated point of E and Y arbitrary infinite subset of E. Then there exists a merotopy Γ on E satisfying:

(a) $cl(\Gamma) = u$,

(b) if we denote by u^* the topology (on E) projectively generated by the ring of all merotopic functions from $\langle E,\Gamma \rangle$ to R, then $x \in u^* Y$.

Proof. Since x is non-isolated, there exist a directed set $\langle A, \leq \rangle$ and a net $\{x_a : a \in A\}$ converging to x with all x_a distinct from x. Since Y is infinite, we may order it by some directed order $\preceq$ such that Y has not the greatest element under $\preceq$.

Define $\mathcal{M} \subset \exp E$ as follows:

$\mathcal{M} = \{M_{a,y} : a \in A, y \in Y\}$, where

$M_{a,y} = \{x_b : b \in A, b \geq a\} \cup \{y' : y' \in Y, y' \succeq y\}$.

Let Γ be a merotopy whose fundamental system is $mer(u) \cup \{\mathcal{M}\}$.

Since $\bigcap \mathfrak{M} = \emptyset$, $cl(\Gamma) = u$.

Let $f \in \mathit{Mer}(\langle E, \Gamma\rangle, R)$. Then there exists a point $z \in \in R$ such that $\mathcal{O}(z)$, its neighborhood system, corefines $f[\mathfrak{M}]$. Obviously $z \in \overline{f[Y]}$ and $\{f(x_a) : a \in A\}$ converges to z. But, according to 2.2, $f: \langle E,u\rangle \longrightarrow R$ is continuous, which implies that $f(x) = z$.

We have proved that for every $f \in \mathit{Mer}(\langle E,\Gamma\rangle, R)$ is true that $f(x) \in \overline{f[Y]}$, thus (u^* is projectively generated by this family) $x \in u^* Y$.

2.4. Convention. Let $\langle E,u\rangle$ be a semi-separated topological space, let Γ be a merotopy on E. The condition "a mapping $f: \langle E,u\rangle \longrightarrow \langle E,u\rangle$ is continuous if and only if the mapping $f: \langle E, \Gamma\rangle \longrightarrow \langle E, \Gamma\rangle$ is merotopic" will be abbreviated to " Γ preserves endomorphisms".

2.5. Theorem. Let $\langle E,u\rangle$ be a semi-separated topological space. Then the merotopy $\Gamma \subset mer(u)$ which induces u is natural if and only if Γ preserves endomorphisms.

Proof. The necessity is obvious.

Sufficiency: Let Γ be an endomorphisms-preserving merotopy, $cl(\Gamma) = u$, $\Gamma \subset mer(u)$. Given arbitrary semi-separated topological space $\mathcal{P} = \langle P,v\rangle$, denote by $\Delta_{\mathcal{P}}$ the finest merotopy on P such that a mapping $f: \langle E, \Gamma\rangle \longrightarrow \langle P, \Delta_{\mathcal{P}}\rangle$ is merotopic whenever $f: \langle E,u\rangle \longrightarrow \langle P,v\rangle$ is continuous. This is always possible since the category **Mer** has inductive generation ([9]). Let $\Gamma_{\mathcal{P}}$ be the finest merotopy on P inducing v (for the description of this merotopy, see [12], p. 252). Let $\mathbb{F}: \mathbf{Top}_{T_1} \longrightarrow \mathbf{Mer}$ be a functor defined by $\mathbb{F}\mathcal{P} = \langle P, \sup(\Delta_{\mathcal{P}}, \Gamma_{\mathcal{P}})\rangle$ for objects, $\mathbb{F}f = f$ for mappings. Then

$\mathbf{F}$ is the desired realization.

I. The merotopy sup($\Delta_{\mathcal{P}}$, $\Gamma_{\mathcal{P}}$) induces v: $\Gamma_{\mathcal{P}}$ induces v, thus cl(sup($\Delta_{\mathcal{P}}$, $\Gamma_{\mathcal{P}}$)) is coarser than v. To show the equality, it suffices to prove that cl($\Delta_{\mathcal{P}}$) is finer than v. Since $\Gamma \subset$ mer(u), every mapping f: $\langle E, \Gamma \rangle \longrightarrow \langle P, \mathrm{mer}(v) \rangle$ is merotopic whenever f: $\langle E, u \rangle \longrightarrow \langle P, v \rangle$ is continuous as a consequence of 2.2. Thus $\Delta_{\mathcal{P}} \subset$ mer(v), because $\Delta_{\mathcal{P}}$ is inductively generated, but this inclusion implies that cl($\Delta_{\mathcal{P}}$) is finer than v.

II. Let $\mathcal{P} = \langle P, v \rangle$, $\mathcal{Q} = \langle Q, w \rangle$ be two semi-separated topological spaces, f a mapping from the set P into the set Q. If f: $\mathbf{F}\mathcal{P} \longrightarrow \mathbf{F}\mathcal{Q}$ is merotopic, then f: $\mathcal{P} \longrightarrow \mathcal{Q}$ is continuous, since by I $\mathbf{F}\mathcal{P}$ ($\mathbf{F}\mathcal{Q}$, resp.) has the merotopy inducing v (w, resp.).

Next, suppose f: $\mathcal{P} \longrightarrow \mathcal{Q}$ to be continuous. Then f: $\langle P, \Gamma_{\mathcal{P}} \rangle \longrightarrow \langle Q, \Gamma_{\mathcal{Q}} \rangle$ is obviously merotopic and if we prove that f: $\langle P, \Delta_{\mathcal{P}} \rangle \longrightarrow \langle Q, \Delta_{\mathcal{Q}} \rangle$ is merotopic, then f: $\mathbf{F}\mathcal{P} \longrightarrow \mathbf{F}\mathcal{Q}$ will be merotopic, too.

Let g: $\langle E, \Gamma \rangle \longrightarrow \langle P, \Delta_{\mathcal{P}} \rangle$ be an arbitrary merotopic mapping. If no such mapping exists, then $\Delta_{\mathcal{P}}$ has a fundamental system $\{\{\{x\}\} : x \in P\}$ and f: $\langle P, \Delta_{\mathcal{P}} \rangle \longrightarrow \langle Q, \Delta_{\mathcal{Q}} \rangle$ is merotopic. If there is at least one such g, then g: $\langle E, u \rangle \longrightarrow$ $\longrightarrow \langle P, v \rangle$ is continuous, thus $f \circ g$: $\langle E, u \rangle \longrightarrow \langle Q, w \rangle$ is continuous and it follows from the definition of $\Delta_{\mathcal{Q}}$ that $f \circ g$: $\langle E, \Gamma \rangle \longrightarrow \langle Q, \Delta_{\mathcal{Q}} \rangle$ is merotopic. Since this holds for every merotopic mapping g: $\langle E, \Gamma \rangle \longrightarrow \langle P, \Delta_{\mathcal{P}} \rangle$ and since a merotopy $\Delta_{\mathcal{P}}$ is inductively generated by the family of all those g's, f is merotopic.

III. Finally, we must show that $\mathbf{F}\langle E,u\rangle = \langle E,\Gamma\rangle$. This is the only point where we need the assumption that Γ preserves endomorphisms. Denote $\mathcal{E} = \langle E,u\rangle$. Since Γ induces u, $\Gamma_{\mathcal{E}} \subset \Gamma$. The merotopy $\Delta_{\mathcal{E}}$ is inductively generated by all continuous mappings $f\colon \langle E,u\rangle \longrightarrow \langle E,u\rangle$, thus $\Delta_{\mathcal{E}} \supset \Gamma$ (the identity mapping is continuous), and the system $\{g[\mathfrak{M}] : \mathfrak{M} \in \Gamma,\ g\colon \langle E,u\rangle \longrightarrow \langle E,u\rangle$ is continuous$\}$ is fundamental for $\Delta_{\mathcal{E}}$. But Γ preserves endomorphisms, thus $g[\mathfrak{M}] \in \Gamma$ whenever $g\colon \mathcal{E} \longrightarrow \mathcal{E}$ is continuous and $\mathfrak{M} \in \Gamma$, hence by the definition of a fundamental system, $\Delta_{\mathcal{E}} \subset \Gamma$.

We have obtained $\Gamma_{\mathcal{E}} \subset \Gamma$, $\Delta_{\mathcal{E}} = \Gamma$, thus $\mathbf{F}\langle E,u\rangle = \langle E, \sup(\Gamma_{\mathcal{E}}, \Delta_{\mathcal{E}})\rangle = \langle E,\Gamma\rangle$ and the proof is finished.

In $\mathbf{Top}_{T_1}$, there are two important full subcategories: The category $\mathbf{P}$ of all coarse semi-separated spaces (i.e. the spaces whose closed subsets are either finite or empty or the whole space) and the category $\mathbf{C}$ of all fine non-discrete spaces (i.e. the non-discrete subspaces of the Čech-Stone compactification of a discrete space, containing precisely one ideal point). It is a well-known fact that every topological semi-separated space $\mathcal{P}$ is projectively (inductively, resp.) generated by the family of all continuous mappings from $\mathcal{P}$ into coarse semi-separated spaces (from fine non-discrete spaces into $\mathcal{P}$, resp.). If we realize that the category $\mathbf{Mer}$ has both the inductive and projective generation, we obtain the following result:

2.6. <u>Theorem</u>. Let $\mathbf{F}\colon \mathbf{P} \longrightarrow \mathbf{Mer}$ ($\mathbf{F}\colon \mathbf{C} \longrightarrow \mathbf{Mer}$, resp.) be a realization. Then $\mathbf{F}$ can be extended into the realization $\mathfrak{G} : \mathbf{Top}_{T_1} \longrightarrow \mathbf{Mer}$.

The proof may be left to the reader.

2.7. Remark. Notice that throughout this paper we have no need to use the assumption cl cl M = cl M. Thus all the results from this chapter will remain valid if we replace "topological" by "closure" everywhere.

3. In [12], the notion of local merotopic character was introduced and some properties of this cardinal invariant were shown. For the sake of completeness we give the definition.

3.1. Definition. Let $\langle E, \Gamma \rangle$ be a (semi-separated) merotopic space, let $x \in E$. Let us define

$\sigma x = \inf \{ \operatorname{card} \Delta : \Delta$ satisfies (o),(i),(ii) below$\}$

(o) $\Delta \subset \Gamma$,

(i) if $\mathcal{M} \in \Delta$, then $x \in \bigcap \mathcal{M}$,

(ii) for every choice $M_{\mathcal{M}} \in \mathcal{M}$, there exists a neighborhood U of x (in $\mathrm{cl}(\Gamma)$) such that $U \subset \bigcup \{ M_{\mathcal{M}} : \mathcal{M} \in \Delta \}$.

The following problem was studied in [12]: Given a closure space $\langle E,u \rangle$, a point $x \in E$ and a cardinal α . Does there exist a merotopy Γ on E inducing u with $\sigma x = \alpha$?

As an example, for $\langle E,u \rangle = [\![0,1]\!]$ and arbitrary $x \in E$ the answer is affirmative whenever $1 \leq \alpha \leq \mathfrak{c}$. But this will never remain true if we are looking for natural merotopies only, since the following holds: Let $\langle E,u \rangle$ be an uncountable separable complete metric space without isolated points, let $x \in E$, let Γ be a natural merotopy for $\langle E,u \rangle$. Then, assuming (CH), either $\sigma x = 1$ or $\sigma x = \mathfrak{c}$ ([12], Theorem 3.9).

This chapter will be devoted to an example that the assumption of completeness cannot be omitted in the theorem above.

3.2. Lemma. Assume (CH). There exist two disjoint subsets P, Q of I ($= [0,1]$) such that the following holds:

(1) $P \cup Q$ cannot be mapped continuously onto I,

(2) if f is continuous real-valued function defined on P and if U is open in I, then $U \cap Q - f[P] \neq \emptyset$,

(2′) if g is continuous real-valued function defined on Q and if V is open in I, then $V \cap P - g[Q] \neq \emptyset$,

(3) both P and Q meet each open subset of I in uncountably many points.

Proof. Let $\mathcal{F}$ be the set of all continuous real-valued functions whose domain is some G_δ-subset of I and whose range is an uncountable subset of I. Then, assuming (CH), we may write $\mathcal{F} = \{f_\alpha : \alpha < \omega_1\}$ and suppose that each $f \in \mathcal{F}$ is listed ω-times.

For $\alpha < \omega_1$, the set $f_\alpha[\mathrm{dom}(f_\alpha)]$ is uncountable, thus, using (CH) once more, we may write $f_\alpha[\mathrm{dom}(f_\alpha)] = \{y_\beta : \beta < \omega_1\}$. Let $E_{\alpha\beta} = f_\alpha^{-1}(y_\beta)$. For $\alpha < \omega_1$, the system $\{E_{\alpha\beta} : \beta < \omega_1\}$ is a pairwise disjoint collection of non-void subsets of I, thus for at most countably many β's the sets $E_{\alpha\beta}$ are non-meager. Denote by S_α the set of all $y_\beta \in f_\alpha[\mathrm{dom}(f_\alpha)]$ such that $E_{\alpha\beta}$ is non-meager; having done this, define $T_\alpha = \bigcup\{S_\iota : \iota \leq \alpha\}$. Finally, let $\{U_n : n < \omega\}$ be an open basis for I and suppose that $U_0 = I$.

The sets P and Q will be defined by a transfinite induction:

$\alpha = 0$: Pick some $E_{0\gamma_0}$ meager and choose two points p_0, $q_0 \in I - (T_0 \cup E_{0\gamma_0})$ such that $p_0 \neq q_0$ and, if $f_0(p_0)$ of $f_0(q_0)$ is defined, then $f_0(p_0) \neq q_0$ and $f_0(q_0) \neq p_0$.

Let $\alpha < \omega_1$ and suppose that p_ι, q_ι, $E_{\iota\gamma_\iota}$ have been defined for all $\iota < \alpha$. Since $\{p_\iota : \iota < \alpha\} \cup \{q_\iota : \iota < \alpha\}$ is countable, there is some $\gamma_\alpha < \omega_1$ such that $E_{\alpha\gamma_\alpha}$ is meager and disjoint with $\{p_\iota : \iota < \alpha\} \cup \{q_\iota : \iota < \alpha\}$.

The following sets

$M_1^\alpha = \bigcup\{f_\iota^{-1}[p_\varkappa] : \iota \leq \varkappa,\ \varkappa, \iota < \alpha\}$

$M_2^\alpha = \bigcup\{f_\iota^{-1}[q_\varkappa] : \iota \leq \varkappa,\ \varkappa, \iota < \alpha\}$

$M_3^\alpha = \{f_\iota(p_\varkappa) : \iota \leq \alpha,\ \varkappa < \alpha\}$

$M_4^\alpha = \{f_\iota(q_\varkappa) : \iota \leq \alpha,\ \varkappa < \alpha\}$

$M_5^\alpha = \{p_\iota : \iota < \alpha\}$

$M_6^\alpha = \{q_\iota : \iota < \alpha\}$

$M_7^\alpha = \bigcup\{E_{\iota\gamma_\iota} : \iota \leq \alpha\}$

are meager: $M_3^\alpha, M_4^\alpha, M_5^\alpha, M_6^\alpha$ are countable and $M_1^\alpha, M_2^\alpha, M_7^\alpha$ are countable unions of meager sets since p_ι, q_ι were never contained in T_ι. Let $M_\alpha = \bigcup\{M_i^\alpha : i = 1,2,\dots,7\}$.

Suppose that $f_\alpha = f$ and that it is exactly the n-th appearance of f in the ordering of $\mathcal{F}$. Then $U_n - (T_\alpha \cup M_\alpha) \neq \emptyset$ and it follows that we can choose $p_\alpha, q_\alpha \in U_n - (T_\alpha \cup M_\alpha)$ such that $p_\alpha \neq q_\alpha$ and, if $f_\alpha(p_\alpha)$ or $f_\alpha(q_\alpha)$ is defined, then $f_\alpha(p_\alpha) \neq q_\alpha$ and $f_\alpha(q_\alpha) \neq p_\alpha$. Since p_α, q_α do not belong to T_α, it is again true that $f_\iota^{-1}[p_\alpha]$ and $f_\iota^{-1}[q_\alpha]$ are meager for each $\iota \leq \alpha$.

It remains to show that $P = \{p_\alpha : \alpha < \omega_1\}$ and $Q = \{q_\alpha : \alpha < \omega_1\}$ are the desired sets.

Suppose $f: P \cup Q \longrightarrow I$ to be continuous. If the range of f is countable, it cannot be the whole I. If the range of f is uncountable, extend f continuously to some G_δ-subset of I; this extension can be found in $\mathcal{F}$, say, on α-th place. From the definition of P and Q we know that $P \cup Q$ is disjoint with $E_{\alpha\gamma_\alpha}$ hence $y_{\gamma_\alpha} \notin f_\alpha[P \cup Q]$ and $f_\alpha[P \cup Q] \supset f[P \cup Q]$. Thus (1) is verified.

The validity of (3) is obvious: If G is an open subset of I, then it contains some base-element U_n, and from the construction of P and Q follows that $\mathrm{card}(U_n \cap P) = \omega_1 = \mathrm{card}(U_n \cap Q)$.

It remains to verify (2), since (2′) is simply the symmetric case. Let f be a continuous function defined on P, let U be an open subset of I. If the range of f is countable, then $U \cap Q - f[P] \neq \emptyset$ by (3). If the range of f is uncountable, denote by g the continuous extension of f to some suitable G_δ-subset of I. The family $\{U_n : n < \omega\}$ is a base for I, so we can find some natural k such that $U_k \quad U$.

Since g belongs to $\mathcal{F}$ and since each member of $\mathcal{F}$ was listed ω-times in the ordering $\{f_\alpha : \alpha < \omega_1\}$, there is some $\lambda < \omega_1$ such that $f_\lambda = g$ and such that this is just the k-th occasion when g appears in $\{f_\alpha : \alpha < \omega_1\}$. The definition of Q implies that $q_\lambda \in Q \cap U_k$ and we are to show that $q_\lambda \notin f_\lambda[P]$: Let $p_\iota \in P$, then

for $\iota = \lambda$, $f_\lambda(p_\lambda) \neq q_\lambda$ by the definition of p_λ, q_λ,

for $\iota < \lambda$, $q_\lambda \neq f_\lambda(p_\iota)$, since $f_\lambda(p_\iota) \in M_4^\lambda$ and $q_\lambda \notin M_\lambda \supset M_4^\lambda$, for $\iota > \lambda$, $p_\iota \notin f_\lambda^{-1}[q_\lambda]$, since $f_\lambda^{-1}[q_\lambda] \subset M_2^\iota \subset M_\iota$ and $p_\iota \notin M_\iota$.

Thus $f(p_\iota) \neq q_\lambda$ for all $\iota < \omega_1$, equivalently, $q_\lambda \in U \cap Q - f[P]$.

3.3. Theorem. Assume (CH), There exists uncountable separable metric space without isolated points $\langle E, u \rangle$ and a natural merotopy Γ for $\langle E, u \rangle$ such that $\sigma x = 2$ for each $x \in E$.

Proof. Let $E = P \cup Q$, where P and Q are the sets from the preceding lemma, with the topology derived from the topology of reals. The topological properties of E follow immediately from (3) of Lemma 3.2.

If $\mathcal{U}(x)$ is the neighborhood system of x in $[0,1]$, let us define

$$\mathfrak{M}_Q(x) = \{U \cap Q \cup \{x\} : U \in \mathcal{U}(x)\},$$
$$\mathfrak{M}_P(x) = \{U \cap P \cup \{x\} : U \in \mathcal{U}(x)\},$$

and let Γ be a merotopy on E, whose fundamental system consists of all $\mathfrak{M}_Q(x)$, $\mathfrak{M}_P(x)$ with $x \in E$ and of all their continuous images under the mappings from $\langle E, u \rangle$ to $\langle E, u \rangle$. Since $\Gamma \subset \mathrm{mer}(u)$ and since Γ preserves endomorphisms, according to 2.5 the merotopy Γ is natural. But from Lemma 3.2 it follows that the neighborhood system of a point $x \in E$ belongs to Γ for no $x \in E$ - see (2),(2′) from the Lemma. Thus $\sigma x \neq 1$, but evidently the system $\Delta = \{\mathfrak{M}_P(x), \mathfrak{M}_Q(x)\}$ is of cardinality 2 and satisfies (0),(i),(ii) from 3.1. Thus $\sigma x = 2$ for each $x \in E$.

4. Let us give another look to the propositions 2.2 and 2.3. If we want to study the natural merotopies, it is clear that the equality $\mathcal{M}er(\langle E,\Gamma\rangle, \langle F,\Delta\rangle) = \mathcal{C}(\langle E,u\rangle,$ $\langle F,v\rangle)$ will be of utmost importance. Proposition 2.2 shows, as a special case, that the implication

$$\Gamma \subset mer(u) \;\&\; cl(\Gamma) = u \Longrightarrow \mathcal{M}er(\langle E,\Gamma\rangle, \langle F,\Delta\rangle) = $$
$$= \mathcal{C}(\langle E,u\rangle, \langle F, cl(\Delta)\rangle)$$

holds whenever $\Delta = mer(cl(\Delta))$ and the Proposition 2.3 indicates that it would not be wise to omit the assumption $\Gamma \subset$ $\subset mer(u)$. What can be said about the reverse implication in the formula above?

We shall give some observations here; the easy proofs are omitted.

4.1. Definition (see [11]). Let P and X be topological spaces. The space X will be called P-regular (P-compact, resp.) if X can be embedded (embedded as a closed subspace, resp.) into some cube P^{α}.

4.2. Proposition. Let P be a semi-separated topological space, $\langle E,u\rangle$ P-regular topological space. If for each merotopy Γ on E is true that $\Gamma \subset mer(u)$ provided that Γ satisfies $cl(\Gamma) = u$ and $\mathcal{M}er(\langle E,\Gamma\rangle, P) = \mathcal{C}(\langle E,u\rangle, P)$, then $\langle E,u\rangle$ is P-compact.

4.3. Corollary. Let $\langle E,u\rangle$ be completely regular Hausdorff and let for every merotopy Γ on E with $cl(\Gamma) = u$ and $\mathcal{M}er(\langle E,\Gamma\rangle, R) = \mathcal{C}(E)$ is true that $\Gamma \subset mer(u)$. Then $\langle E,u\rangle$ is realcompact.

4.4. Proposition. Let $\langle E,u\rangle$ be a completely regular Hausdorff topological space. Then $\langle E,u\rangle$ is compact if and only if for every merotopy Γ on E such that $cl(\Gamma) = u$ and

$\mathcal{Mer}(\langle E, \Gamma \rangle, [\![0,1]\!]) = \mathcal{C}(\langle E,u \rangle, [\![0,1]\!])$ is true that $\Gamma \subset \mathrm{mer}(u)$.

4.5. Corollary. Let $\langle E,u \rangle$ be a completely regular Hausdorff space, Γ merotopy on E, $\mathrm{cl}(\Gamma) = u$. Denote the Čech-Stone compactification $\beta\langle E,u \rangle$ as $\langle \tilde{E},\tilde{u} \rangle$. Then the following are equivalent:

(a) $\mathcal{Mer}(\langle E, \Gamma \rangle, [\![0,1]\!]) = \mathcal{C}(\langle E,u \rangle, [\![0,1]\!])$

(b) $\Gamma \subset \mathrm{mer}(\tilde{u}) \cap \exp\exp E$.

4.6. Remark. Compare 4.4 and 4,2. It may seem that in 4.2 the reverse implication should be valid, too. This is not true, not even in the case P = R (realcompact spaces).

References

[1] BENTLEY H.L.: Normal nearness spaces, preprint.

[2] BENTLEY H.L., HERRLICH H., ROBERTSON W.A.: Convenient categories for topologists, Comment. Math. Univ. Carolinae 17(1976), 207-227.

[3] ČECH E.: Topological spaces, Academia, Prague 1966.

[4] HERRLICH H.: A concept of nearness, Gen. Top. and Appl. 4(1974), 191-212.

[5] HERRLICH H.: Topological structures, Math. Centre Tracts 52,1974, 59-122.

[6] HUNSAKER W.N., SHARMA P.L.: Nearness spaces compatible with a topological space, Arch. Math. (Basel),25 (1974), 172-178.

[7] ISBELL J.R.: Uniform spaces, Providence 1964.

[8] KATĚTOV M.: Convergence structures, Gen. Top. and its Rel. to Modern Anal. and Alg. II, Proc. 2nd Prague Top. Symp.,1966, pp. 207-216.

[9] KATĚTOV M.: On continuity structures and spaces of mappings, Comment. Math. Univ. Carolinae 6(1965), 257-278.

[10] KATĚTOV M.: Prostranstva, opredeljaemyje zadanijem semejstva centrirovanych sistem, Uspechi mat. nauk 5(191)(1976), 95-107.

[11] MRÓWKA S.: On E-compact spaces II, Bull. Acad. Polon. Sci. 14(1966), 597-605.

[12] SIMON P.: On local merotopic character, Comment. Math. Univ. Carolinae 12(1971), 249-270.

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