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FIXED POINT THEOREMS FOR PSEUDOCONTRACTIVE MAPPINGS AND A
COUNTEREXAMPLE FOR COMPACT MAPS

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Abstract: We give examples for open bounded starshaped sets in all normable spaces of dimension at least 3 whose closures have not the fixed point property for compact self-mappings. Using a special convergence theorem we extend fixed point theorems for pseudocontractive mappings (including non-expansive mappings) which are known for Hilbert spaces.

Key words: Fixed points, starshaped sets, compact mappings, pseudocontractive and nonexpansive mappings, duality mappings.

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0. Introduction. It is well-known that the Brouwer fixed point theorem need not be true for compact starshaped subsets of a finite-dimensional space (see [12],[13],[14],[15]). However, the counterexamples given with respect to this problem are essentially boundary sets in the underlying space. In this paper we shall give even an example of a compact starshaped subset of three-dimensional space \mathbb{R}^3 which is the closure of an open starshaped set but has not the fixed point property for continuous maps. Moreover we shall present a theorem on (strong) convergence for Banach spaces having a weakly continuous duality mapping. Then we have both a generalization of a corresponding result for Hilbert spaces which is due to

M.G. Crandall and A. Pazy [3] and a couple of applications to the fixed point theory of pseudocontractive and nonexpansive mappings in Banach spaces possessing a weakly continuous duality mapping (for Hilbert spaces some of the results are known, see [12],[13],[14],[15],[16],[18]).

For a normed linear space $(E, \|\cdot\|)$ E^* denotes the strong dual space of $(E, \|\cdot\|)$ and for a subset X of E let \bar{X} , $\text{int}(X)$, ∂X denote the closure of X , the interior of X and the boundary of X respectively. $X \subset E$ is said to be starshaped iff there exists $x_0 \in X$ such that $tx + (1-t)x_0 \in X$ for $x \in X$ and $t \in [0,1]$. For $f: X \rightarrow E$ we define $\text{Fix}(f) := \{x \mid x \in X \wedge f(x) = x\}$.

1. A counterexample In this section we give an example for an open bounded and starshaped subset of \mathbb{R}^3 whose closure has not the fixed point property for continuous self-mappings. Moreover we discuss some consequences of this result to \mathbb{R}^n ($n \geq 3$) and other spaces.

For the definition of the set in \mathbb{R}^3 described below we use a construction and a hint of J.M. Lysko [10].

Theorem 1.1 There exist $X \subset \mathbb{R}^3$ and $f \in C(\bar{X}, \bar{X})$ such that

- (i) X is open bounded and starshaped,
- (ii) $\text{Fix}(f) = \emptyset$.

Proof: (1) Let $p: [\frac{1}{2}, 1) \rightarrow \mathbb{R}^+$ be defined as follows:

$$\forall_{\substack{n \in \mathbb{N} \\ n \geq 2}} \forall_{t \in [0,1]} p((1-t)(1 - \frac{1}{n}) + t(1 - \frac{1}{n+1})) := (n-2+t) \cdot \pi$$

Define $X, Y \subset \mathbb{R}^3$ respectively by

$$X := \{(r \cos \varphi, r \sin \varphi, z) \mid r \in [0,1), \varphi \in \mathbb{R}^+, z \in (-1,1)\} \cup$$

$$\{(|z| r \cos \varphi, |z| r \sin \varphi, z) \mid r \in (\frac{1}{2}, 1), 1 \leq |z| < 2, p(r) < \varphi < p(r) + 1\}$$

and

$$\begin{aligned}
 Y := & \{(r \cos \varphi, r \sin \varphi, z) \mid r \in [0, 1], \varphi \in \mathbb{R}^+, z \in [-1, 1]\} \cup \\
 & \{|z| \cos \varphi, |z| \sin \varphi, z \mid 1 \leq |z| \leq 2, \varphi \in \mathbb{R}^+\} \cup \\
 & \{|z| r \cos \varphi, |z| r \sin \varphi, z \mid r \in [\frac{1}{2}, 1), 1 \leq |z| \leq 2, \\
 & \quad p(r) \leq \varphi \leq p(r) + 1\}.
 \end{aligned}$$

By a straightforward but somewhat lengthy computation we obtain: X is a bounded open and starshaped (with respect to the origin) subset of \mathbb{R}^3 such that $\bar{X} = Y$. Now let $H: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be defined as follows:

$$\begin{aligned}
 \text{If } (x, y, z) \in \mathbb{R}^3 \text{ and } |z| \leq 1 \text{ then } H(x, y, z) &:= (x, y, z); \\
 \text{if } (x, y, z) \in \mathbb{R}^3 \text{ and } |z| > 1 \text{ then } H(x, y, z) &:= \left(\frac{x}{|z|}, \frac{y}{|z|}, z \right).
 \end{aligned}$$

H clearly is a homeomorphism. Let $K := H[Y]$. Thus

$$\begin{aligned}
 K = & \{(r \cos \varphi, r \sin \varphi, z) \mid r \in [0, 1], \varphi \in \mathbb{R}^+, |z| \leq 1\} \cup \\
 & \{(\cos \varphi, \sin \varphi, z) \mid \varphi \in \mathbb{R}^+, 1 \leq |z| \leq 2\} \cup \\
 & \{(r \cos \varphi, r \sin \varphi, z) \mid r \in [\frac{1}{2}, 1), 1 \leq |z| \leq 2, p(r) \leq \\
 & \quad \leq \varphi \leq p(r) + 1\}.
 \end{aligned}$$

Clearly it is enough to prove the existence of a $g \in C(K, K)$ such that $\text{Fix}(g) = \emptyset$.

Let $R: K \rightarrow \mathbb{R}^3$ be defined as follows:

$$\begin{aligned}
 \text{If } (x, y, z) \in K \text{ and } z \leq 1 \text{ then } R(x, y, z) &:= (x, y, 1); \\
 \text{if } (x, y, z) \in K \text{ and } x^2 + y^2 = 1 \text{ and } z \geq 1 \text{ then } R(x, y, z) &:= (x, y, z); \\
 \text{if } (x, y, z) \in K \text{ and } x = r \cos \varphi, y = r \sin \varphi \text{ and } r \in [\frac{1}{2}, 1), z \geq 1 \\
 \text{and } p(r) \leq \varphi \leq p(r) + 1 \text{ then} \\
 R(x, y, z) := & \begin{cases} (p^{-1}(\varphi) \cos \varphi, p^{-1}(\varphi) \sin \varphi, z + r - p^{-1}(\varphi)) & \text{if } z + r - p^{-1}(\varphi) \geq 1 \\ ((z - 1 + r) \cos \varphi, (z - 1 + r) \sin \varphi, 1) & \text{if } z + r - p^{-1}(\varphi) < 1. \end{cases}
 \end{aligned}$$

For $z \in [1, 2]$ and $r \in [\frac{1}{2}, 1)$ such that $p(r) \leq \varphi \leq p(r) + 1$

we have $z + r - p^{-1}(\varphi) \leq z \leq 2$ and conversely $z + r - p^{-1}(\varphi) \leq 1$ implies

$$\frac{1}{2} \leq r \leq z - 1 + r \leq p^{-1}(\varphi) < 1 \text{ thus}$$

$$\begin{aligned} R[K] \subset K' := & \{(r \cos \varphi, r \sin \varphi, 1) \mid r \in [0, 1], \varphi \in \mathbb{R}^+\} \cup \\ & \{(\cos \varphi, \sin \varphi, z) \mid z \in [1, 2], \varphi \in \mathbb{R}^+\} \cup \\ & \{(p^{-1}(\varphi) \cos \varphi, p^{-1}(\varphi) \sin \varphi, z) \mid z \in [1, 2], \\ & \varphi \in \mathbb{R}^+\} \subset K. \end{aligned}$$

Moreover we have $R|_{K'} = \text{Id}_{K'}$, thus $R[K'] = K'$.

Obviously R is continuous (it clearly suffices to verify this for points $(\cos \varphi_0, \sin \varphi_0, z_0)$ with $\varphi_0 \in \mathbb{R}^+$ and $z_0 \in [1, 2]$).

We have: R is a retraction from K onto $K' \subset K$. Therefore it is enough to search for a map $h \in C(K, K')$ such that

$\text{Fix}(h) = \emptyset$ (then $g := h \circ R \in C(K, K)$ and $\text{Fix}(g) = \emptyset$). Let $h: K' \rightarrow \mathbb{R}^3$ be defined as illustrated in the schedule on page 296.

Remark 1.2. (i) Let $n \in \mathbb{N}$, let $X \subset \mathbb{R}^n$ be open bounded and starshaped with respect to $b \in X$. Suppose $f \in C(\bar{X}, \bar{X})$ such that $\text{Fix}(f) = \emptyset$. Let $j: \mathbb{R}^n \rightarrow \mathbb{R}^{n+1}$ be the natural embedding. Then the "open cone over X "

$$Y := \{(z_1, \dots, z_{n+1}) \in \mathbb{R}^{n+1} \mid z_{n+1} \in (0, 1) \wedge \frac{1}{1-z_{n+1}}(z_1, \dots, z_n) \in \bar{X}\}$$

is an open bounded and starshaped subset of \mathbb{R}^{n+1} and $g \in C(\bar{Y}, \bar{Y})$ defined by $g((1-t)(0, \dots, 0, 1) + tj(x)) := j(f((1-t)b + tx))$ for $t \in [0, 1]$, $x \in \bar{X}$ has no fixed points (compare [8]). Thus we obtain by Theorem 1.1:

$$\forall_{\substack{n \in \mathbb{N} \\ n \geq 3}} \exists_{\substack{X \subset \mathbb{R}^n \\ X \text{ open bounded starshaped}}} \exists_{f \in C(\bar{X}, \bar{X})} X \text{ open bounded starshaped} \wedge \text{Fix}(f) = \emptyset.$$

(ii) Let (E, \mathcal{T}) be a separated locally convex topological linear space of dimension at least 3. Then there exist an

open starshaped set K and a compact map $g \in C(\bar{K}, \bar{K})$ such that $\text{Fix}(g) = \emptyset$. Indeed, let F be a 3-dimensional linear subspace of E . As there are linear homeomorphisms between \mathbb{R}^3 and F , there is an open bounded starshaped set (with respect to the origin) X in F and $f \in C(\bar{X}, \bar{X})$ with $\text{Fix}(f) = \emptyset$. Let P be any continuous linear projection of E onto F (F is a complementary set). Set $K := P^{-1}[X]$ and $g := f \circ P|_{\bar{K}}$. Then it is clear that K is open and starshaped and g is a compact map such that $\text{Fix}(g) = \emptyset$. If, in addition, (E, \mathcal{Z}) is normable, K may be taken bounded.

2. Fixed point theorems for pseudocontractive mappings.

A convergence theorem due to M.G. Grandall and A. Pazy [3] implies several fixed point theorems for continuous pseudocontractive and especially for nonexpansive mappings in Hilbert space (see [12],[13],[14],[15],[16],[18]). In the present note we establish a variant of that theorem which guarantees that most of these results are valid for a more general class of spaces.

Definition 2.1 (i) $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is said to be a gauge function : $\iff \mu$ is continuous and strictly monotone, $\mu(0) = 0$, $\lim_{t \rightarrow \infty} \mu(t) = \infty$.

(ii) Let $(E, \|\cdot\|)$ be a real normed space, $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function and $J : E \rightarrow E^*$. J is called a duality mapping with respect to $\mu : \iff \forall x \in E \quad J(x)(x) = \|x\| \cdot \mu(\|x\|) \wedge \|J(x)\| = \mu(\|x\|)$.

(iii) $((E, \|\cdot\|), \mu, J)$ satisfies $(*) : \iff (E, \|\cdot\|)$ is a reflexive real normed space, $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a gauge

function and $J: E \rightarrow E^*$ is a weakly sequentially continuous duality mapping with respect to μ^1 .

Remark 2.2 (i) Let $(E, \|\cdot\|)$ be a real normed space, let $\mu: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function. Then the Hahn-Banach theorem implies the existence of a duality mapping $J: E \rightarrow E^*$ with respect to μ .

(ii) Let $(E, (\cdot, \cdot))$ be a real Hilbert space. Define $J: E \rightarrow E^*$ by $J(x)(y) := (y, x)$ and $\|\cdot\|: E \rightarrow \mathbb{R}$ by $\|x\| := (x, x)^{1/2}$. Then $((E, \|\cdot\|), \text{Id}_{\mathbb{R}^+}, J)$ satisfies (*).

(iii) Let $p, q \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then we identify ℓ_p^*, ℓ_q in the usual manner. Define $J: \ell_p \rightarrow \ell_q$ by $J((x_j)_{j \in \mathbb{N}}) := (|x_j|^{p-1} \text{sign } x_j)_{j \in \mathbb{N}}$ and $\mu: \mathbb{R}^+ \rightarrow \mathbb{R}$ by $\mu(t) := t^{p-1}$. Then $((\ell_p, \|\cdot\|), \mu, J)$ satisfies (*). (See [2]).

Definition 2.3 Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq X \subset E$, $f: X \rightarrow E$.

(i) f is said to be nonexpansive : $\iff \forall_{x, y \in X} \|f(x) - f(y)\| \leq \|x - y\|$

(ii) f is said to be pseudocontractive : \iff

$$\forall_{x, y \in X} \forall_{r \in \mathbb{R}} \|x - y\| \leq \|(1+r)(x-y) - r(f(x) - f(y))\|$$

Remark 2.4 Let $(E, \|\cdot\|)$ be a real normed space, $\emptyset \neq X \subset E$, $f: X \rightarrow E$.

(i) If f is nonexpansive then f is pseudocontractive.

(ii) If there is a uniquely determined duality mapping

 1) This implies that E^* is strictly-convex (see [61] and consequently J is unique.

$J: E \rightarrow E^*$ with respect to some gauge function then we have: f pseudocontractive $\iff \forall x, y \in X J(x - y)(f(x) - f(y)) \leq J(x - y)(x - y)$ (see [7]).

The announced convergence theorem is

Lemma 2.5 ([12]) Let $(E, \|\cdot\|)$ be a real normed space admitting a weakly sequentially continuous duality mapping $J: E \rightarrow E^*$ with respect to some gauge function μ , let $(x_n) \in E^{\mathbb{Z}^+}$, $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that
 (i) $\lim(x_n) = x_0$ (weakly), (ii) $\lim(r_n) = 0$,
 (iii) $\forall n, m \in \mathbb{N} J(x_n - x_m)(r_n x_n - r_m x_m) \leq 0$

Then $\lim(x_n) = x_0$ (strongly).

Proof: We have for $n \in \mathbb{N}$ $\lim_m(J(x_n - x_m)) = J(x_n - x_0)$ (weakly), $\lim_m(-r_n x_n + r_m x_m) = -r_n x_n$ (strongly). This together with (iii) implies: $J(x_n - x_0)(-r_n x_n) = \lim_m(J(x_n - x_m)(-r_n x_n + r_m x_m)) \geq 0$, hence $J(x_n - x_0)(-x_0) = J(x_n - x_0)(x_n - x_0) + J(x_n - x_0)(-x_n) \geq J(x_n - x_0)(x_n - x_0) = \|x_n - x_0\| \cdot \mu(\|x_n - x_0\|)$. Because of $\lim(J(x_n - x_0)(-x_0)) = 0$ we get $\lim(\|x_n - x_0\|) = 0$. As an evident consequence of Lemma 2.5 we get

Lemma 2.6 Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, $(x_n) \in E^{\mathbb{N}}$, $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that (x_n) is bounded, $\lim(r_n) = 0$ and $\forall n, m \in \mathbb{N} J(x_n - x_m)(r_n x_n - r_m x_m) \leq 0$.

Then there is a subsequence (y_n) of (x_n) and $y \in E$ such that $\lim(y_n) = y$ (strongly).

Lemma 2.6 implies the following fixed point theorem for continuous pseudocontractive mappings:

Lemma 2.7 (See [16]) Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $\emptyset \neq X \subset E$ be closed and $f: X \rightarrow E$ be continuous and pseudocontractive, let $(x_n) \in X^{\mathbb{N}}$, $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ such that

(i) (x_n) is bounded, (ii) $\lim(\lambda_n) = 1$,

(iii) $\forall_{n \in \mathbb{N}} x_n = \lambda_n f(x_n)$

Then f has a fixed point.

Proof: We define $(r_n) \in (0, \infty)^{\mathbb{N}}$ by $r_n := \frac{1}{\lambda_n} - 1$. As f is pseudocontractive we get for $n, m \in \mathbb{N}$:

$-J(x_n - x_m)(r_n x_n - r_m x_m) = J(x_n - x_m)(x_n - x_m - f(x_n) + f(x_m)) \geq 0$ (see Remark 2.4 (ii)). Lemma 2.6 guarantees $y \in E$ and a subsequence (y_n) of (x_n) such that $\lim(y_n) = y$ (strongly).

Then $y \in X$ and because of $\lim(y_n - f(y_n)) = 0$ and continuity of f we get: $f(y) = y$.

The following theorems are applications of Lemma 2.7. For Hilbert spaces and Lipschitzian pseudocontractive mappings the theorems 2.8 and 2.9 are proved in [16] and for merely continuous pseudocontractive mappings they are proved in [18].

Theorem 2.8 Let $((E, \|\cdot\|), \mu, J)$ satisfy $(*)$, let $X \subset E$ be a closed neighborhood of the origin and $f: X \rightarrow E$ be continuous and pseudocontractive such that $f[X]$ is bounded and

$$(LS) \quad \forall_{x \in \partial X} \quad \forall_{\lambda \in \mathbb{R}} \quad \lambda f(x) = \lambda x \implies \lambda \leq 1$$

Then f has a fixed point.

Proof: Choose $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$ $\lambda_n f$ is continuous and strictly pseudocontractive with

$$\forall_{x \in \partial X} \quad \forall_{\lambda \in \mathbb{R}} \quad (\lambda_n f)(x) = \lambda x \implies \lambda \leq 1.$$

By a theorem of R. Schöneberg [18] there is $(x_n) \in X^{\mathbb{N}}$ such

that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$. According to Lemma 2.7 we are done.

Theorem 2.9 Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$, let $X \subset E$ be a closed and symmetric neighborhood of the origin and $f: X \rightarrow E$ be continuous and pseudocontractive such that $f[X]$ is bounded and $\forall x \in \partial X \quad f(-x) = -f(x)$.

Then f has a fixed point.

Proof: For $x \in \partial X$ we have $J(2x)(2f(x)) = J(x - (-x)) \cdot (f(x) - f(-x)) \leq J(x - (-x))(x - (-x)) = J(2x)(2x)$. Thus f satisfies condition (IS) of Theorem 2.8.

Lemma 2.10 Let E be a topological linear space and $X \subset E$ be starshaped with respect to the origin. Assume $f: \bar{X} \rightarrow E$ such that

$$(R) \quad \forall x \in \partial X \quad \exists \lambda > 0 \quad \forall t \in (0, \lambda] \quad (1+t)x - tf(x) \notin \bar{X}.$$

Then $\forall x \in \partial X \quad \forall \lambda \in \mathbb{R} \quad f(x) = \lambda x \implies \lambda \leq 1$.

Proof: Let $x \in \partial X$, $\lambda \in \mathbb{R}$ and $f(x) = \lambda x$. Suppose $\lambda > 1$. Choose $\tilde{\lambda} > 0$ such that $(1+t)x - tf(x) \notin \bar{X}$ for $t \in (0, \tilde{\lambda}]$ and choose $t \in (0, \tilde{\lambda}]$ such that $(\lambda - 1)t \in (0, 1]$. Then we have $(1+t)x - tf(x) = (1 - (\lambda - 1)t)x \in \bar{X}$ since \bar{X} is starshaped with respect to the origin, too. This contradicts (R), thus $\lambda \leq 1$.

Observing Lemma 2.10 and Theorem 2.8 we obtain

Theorem 2.11 Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$. Suppose $X \subset E$ is closed and starshaped with respect to $0 \in \text{int}(X)$ and $f: X \rightarrow E$ is continuous and pseudocontractive such that $f[X]$ is bounded and

$$(R) \quad \forall x \in \partial X \quad \exists \lambda > 0 \quad \forall t \in (0, \lambda) \quad (1+t)x - tf(x) \notin \bar{X}.$$

Then f has a fixed point.

Remark 2.12 Lemma 2.10 shows that H. Rothe's fixed point theorem for compact maps in [17] is only a special case of the general Leray-Schauder fixed point theorem for compact maps.

Theorem 2.13 Let $(E, \|\cdot\|), (\mu, J)$ satisfy $(*)$. Suppose $X \subset E$ is a closed bounded and symmetric neighborhood of the origin, $f: X \rightarrow E$ is continuous and pseudocontractive such that $f[X]$ is bounded and

$$(A) \quad \exists \varepsilon > 0 \quad \forall x \in \partial X \quad \|f(x) + f(-x)\|^2 - \|2x - f(x) + f(-x)\|^2 \leq 4(1-\varepsilon) \|x - f(x)\| \cdot \|x + f(-x)\|$$

and

$$(B) \quad \inf \{ \|x - f(x)\| \mid x \in \partial X \} > 0$$

Then f has a fixed point.

Proof: Let $\varepsilon > 0$ be chosen according to (A). Let $M > 0$ such that $\|f(x)\| < M, \|x\| < M$ for $x \in X, r := \inf \{ \|x - f(x)\| \mid x \in \partial X \},$ let $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ such that $\lim(\lambda_n) = 1$ and $(1 - \lambda_n) \cdot 9M^2 < \varepsilon \cdot r^2$ for $n \in \mathbb{N}$. Then we have for $x \in \partial X, n \in \mathbb{N}$:

$$\begin{aligned} & \frac{1}{4} \|\lambda_n f(x) + \lambda_n f(-x)\|^2 - \frac{1}{4} \|2x - \lambda_n f(x) + \lambda_n f(-x)\|^2 \\ & \leq \frac{1}{4} \|f(x) + f(-x)\|^2 - \frac{1}{4} \|2x - f(x) + f(-x)\|^2 + \\ & \quad + (1 - \lambda_n) 4M^2 \leq (1 - \varepsilon) \|x - f(x)\| \cdot \|x + f(-x)\| + \\ & \quad + (1 - \lambda_n) \cdot 4M^2 \leq \|x - f(x)\| \cdot \|x + f(-x)\| - \varepsilon \cdot r^2 + \\ & \quad + (1 - \lambda_n) \cdot 4M^2 \leq (\|x - \lambda_n f(x)\| + (1 - \lambda_n)M) \cdot \end{aligned}$$

$\cdot (\|x + \lambda_n f(-x)\| + (1 - \lambda_n)M) - \epsilon \cdot r^2 + (1 - \lambda_n) \cdot 4M^2$
 $< \|x - \lambda_n f(x)\| \cdot \|x + \lambda_n f(-x)\|$, hence
 $x - \lambda_n f(x) = \mu(-x - \lambda_n f(-x))$ for $n \in \mathbb{N}$, $x \in \partial X$,
 $\mu \in (0, 1]$.

By a theorem of R. Schöneberg [18] we obtain a sequence (x_n) such that $x_n = \lambda_n f(x_n)$. Hence f has a fixed point by Lemma 2.7.

Remark 2.14 (1) In the case of a Hilbert space $(E, (\cdot, \cdot))$ the condition (A) of Theorem 2.13 is equivalent to

$$\exists \epsilon > 0 \quad \forall x \in \partial X \left(\frac{\|x - f(x)\|}{\|x - f(x)\|}, \frac{\| -x - f(-x) \|}{\| -x - f(-x) \|} \right) \leq 1 - \epsilon$$

(ii) For nonexpansive mappings we get the following

Theorem: Let $(E, \|\cdot\|)$ be a uniformly convex space. Suppose $X \subset E$ is a closed bounded convex symmetric neighborhood of the origin and let $f: X \rightarrow E$ be nonexpansive such that (A) of Theorem 2.13 is fulfilled.

Then f has a fixed point.

The proof is based upon the fact that $\text{Id}_X - f$ is demi-closed.

Theorem 2.15 (see [18]) Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$, let $X \subset E$ be closed and bounded with $\text{int}(X) \neq \emptyset$. Suppose $f: X \rightarrow E$ is continuous and pseudocontractive such that $f|_X$ is bounded and $\exists z \in X \quad \forall x \in \partial X \quad \|z - f(z)\| < \|x - f(x)\|$ ("minimum principle")

Then f has a fixed point.

Proof: Theorem 1 of [9] implies $\inf \{ \|x - f(x)\| \mid x \in X \} = 0$. Without loss of generality we may assume that

$a := \inf \{ \|x - f(x)\| \mid x \in \partial X \} > 0$ and that there exists $z \in X$ such that $\|z - f(z)\| < a$. Moreover we may assume $\|f(0)\| < a$. Choose $(r_n) \in (0, \infty)^{\mathbb{N}}$ such that $\lim(r_n) = 0$ and $r_n \|x\| + \|f(0)\| < a$ for $n \in \mathbb{N}$ and $x \in X$. Define $T_n: X \rightarrow E$ by $T_n := (1 + r_n)\text{Id}_X - f$, let $n \in \mathbb{N}$. Then we have for $x, y \in X$:

$$\mu(\|x - y\|) \|T_n(x) - T_n(y)\| \geq J(x - y)(T_n(x) - T_n(y)) \geq J(x - y)(r_n x - r_n y) = r_n \mu(\|x - y\|) \|x - y\|,$$
 hence $(0) \|T_n(x) - T_n(y)\| \geq r_n \|x - y\|$, and for $x \in \partial X$: $\|T_n(0)\| = \|f(0)\| < \|x - f(x)\| - r_n \|x\| \leq \|T_n(x)\|$. Theorem 1 of [9] implies: $0 \in \overline{T_n[X]}$, and because of (0): $0 \in T_n[X]$. That means: There is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \frac{1}{1+r_n} f(x_n)$ for $n \in \mathbb{N}$. Lemma 2.7 completes the proof.

Remark 2.16 From Theorem 1 of [9] we learn that Theorem 2.15 remains true if the assumption " $((E, \|\cdot\|), (\mu, J))$ satisfies $(*)$ " is replaced by " $(E, \|\cdot\|)$ is a Banach space and X has the fixed point property with respect to nonexpansive self-mappings".

Lemma 2.17 Let $(E, \|\cdot\|)$ be a normed space. Suppose $X \subset E$ is closed and starshaped with respect to the origin, $\lambda \in (0, 1)$ and $f: X \rightarrow E$ such that

$$\forall x \in \partial X \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{1}{t} d((1-t)x + tf(x), X) = 0$$

Then
$$\forall x \in \partial X \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{1}{t} d((1-t)x + t \cdot \lambda f(x), X) = 0$$

Theorem 2.18 Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$. Suppose $X \subset E$ is closed, bounded and starshaped and $f: X \rightarrow E$ is continuous and pseudocontractive such that

$$\forall x \in \partial X \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{1}{t} d((1-t)x + tf(x), X) = 0$$

Then f has a fixed point.

Proof: Define $\tilde{J}: E \rightarrow E^*$ by $J(0) := 0$, $\tilde{J}(x) := \frac{\|x\|}{\mu(\|x\|)} J(x)$ for $x \in E \setminus \{0\}$. \tilde{J} is the (uniquely determined) duality mapping with respect to $\text{Id}_{\mathbb{R}^+}$. Without loss of generality we assume X to be starshaped with respect to the origin. Choose $(\lambda_n) \in (0,1)^{\mathbb{N}}$ such that $\lim(\lambda_n) = 1$. Then we have for $n \in \mathbb{N}$:

(i) $\lambda_n f$ is continuous

(ii) $\tilde{J}(x - y)(\lambda_n f(x) - \lambda_n f(y)) \leq \lambda_n \|x - y\|^2$

(iii) $\forall x \in \partial X \quad \lim_{\substack{t \rightarrow 0 \\ t > 0}} \frac{1}{t} d((1-t)x + t \cdot \lambda_n f(x), X) = 0$ (Lemma

2.17)

A theorem of R.H. Martin [11] and K. Deimling [4] implies the existence of $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and Lemma 2.7 completes the proof.

Remark 2.19 For lipschitzian pseudocontractive mappings in Hilbert spaces Theorem 2.18 was proved by D. Göhde [5]. If X is assumed to be convex it was shown in [16] that the assumption "f be lipschitzian" can be dropped.

Theorem 2.20 Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ is closed and bounded and $f: X \rightarrow E$ is nonexpansive such that $\text{co } f[\partial X] \subset X$.

Then f has a fixed point.

Proof: Without loss of generality $0 \in f[\partial X]$. Let $(\lambda_n) \in (0,1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$. For $n \in \mathbb{N}$, $x \in \partial X$ we have: $\lambda_n f$ is a Banach-contraction and $(\lambda_n f)(x) \in \text{co } f[\partial X] \subset X$, thus $(\lambda_n f)[\partial X] \subset X$.

According to a theorem due to N.A. Assad [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = \lambda_n f(x_n)$ for $n \in \mathbb{N}$, and by Lemma 2.7 we obtain the conclusion.

Lemma 2.21 Let $(E, \|\cdot\|)$ be a normed space, $J: E \rightarrow E^*$ be a duality mapping with respect to some gauge function $\mu: \mathbb{R}^+ \rightarrow \mathbb{R}$. Suppose $x, z \in E$, $M > 0$, $\|x\| \geq 3M$, $\|z\| < M$.

Then (i) $0 < \mu(2M) \leq \|J(x - z)\| \leq \mu(\|x\| + M)$

$$(ii) \quad J(x - z)(x) \geq \mu(2M) \cdot M.$$

Proof: (i) $\|J(x - z)\| = \mu(\|x - z\|) \geq \mu(\|x\| - \|z\|) \geq \mu(3M - M)$, $\|J(x - z)\| \leq \mu(\|x\| + \|z\|) \leq \mu(\|x\| + M)$

$$(ii) \quad J(x - z)(x) = J(x - z)(x - z) + J(x - z)(z)$$

$$\geq \|J(x - z)\| (\|x - z\| - \|z\|) \geq \mu(2M) \cdot M.$$

Lemma 2.22 Let $(E, \|\cdot\|)$ be a normed space, $M, r > 0$, $x \in E$, $\|x\| \geq 3M$, $\emptyset \neq S \subset E$ and suppose $\|z\| < M$ for $z \in S$.

Then $\inf_{z \in S} \|(1+r)x - z\| > \inf_{z \in S} \|x - z\|$

Proof: Let $J: E \rightarrow E^*$ be a duality mapping with respect to $\text{Id}_{\mathbb{R}^+}$. Then we have for $z \in S$: $\|J(x - z)\| \|(1+r)x - z\| \geq J(x - z)(x - z) + J(x - z)(rx) = \|J(x - z)\| \|x - z\| + r J(x - z)(x) \geq \|J(x - z)\| (\|x - z\| + r \cdot 2 \cdot M \cdot M)$, and from

$$\|(1+r)x - z\| \geq \|x - z\| + \frac{2rM^2}{\|J(x-z)\|} \geq \|x - z\| + \frac{2rM^2}{\|x\| + M} \text{ and } \frac{2rM^2}{\|x\| + M} > 0 \text{ the conclusion follows.}$$

Lemma 2.23 Let $(E, \|\cdot\|)$ be a normed space, $\emptyset \neq X \subset E$ and $f: X \rightarrow E$ be nonexpansive. Suppose $x_0 \in X$ such that $(f^n(x_0))_{n \in \mathbb{Z}^+}$ is bounded. Finally let $(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ such that

$$f(x_n) = (1 + \frac{1}{n})x_n \text{ for } n \in \mathbb{N}$$

Then $(x_n)_{n \in \mathbb{N}}$ is bounded.

Proof: Define $S := \{f^n(x_0) \mid n \in \mathbb{Z}^+\}$. Obviously $S \neq \emptyset$. Choose $M > 0$ such that $\|z\| < M$ for $z \in S$. We claim $\|x_n\| < 3M$ for $n \in \mathbb{N}$. Otherwise we would have $\|x_n\| \geq 3M$ for a suitable $n \in \mathbb{N}$, hence by Lemma 2.20 $\inf_{z \in S} \|f(x_n) - z\| = \inf_{z \in S} \|(1 + \frac{1}{n})x_n - z\| > \inf_{z \in S} \|x_n - z\|$ for this n . Choose $y \in S$ with $\|x_n - y\| < \inf_{z \in S} \|f(x_n) - z\|$. Observing $f(y) \in S$ we get $\|f(x_n) - f(y)\| \geq \inf_{z \in S} \|f(x_n) - z\| > \|x_n - y\|$ and this is a contradiction to the nonexpansiveness of f .

Theorem 2.24 Let $((E, \|\cdot\|), (\mu, J))$ satisfy $(*)$. Suppose $\emptyset \neq X \subset E$ is closed and starshaped and $f: X \rightarrow E$ is nonexpansive such that:

- (i) $f[\partial X] \subset X$
- (ii) there is $x_0 \in X$ such that $(f^n(x_0))_{n \in \mathbb{Z}^+}$ is bounded.

Then f has a fixed point.

Proof: Without loss of generality let X be starshaped with respect to the origin. Then for $n \in \mathbb{N}$ $(1 - \frac{1}{n+1})f$ is a Banach-contraction with $(1 - \frac{1}{n+1})f[\partial X] \subset X$. By a theorem of N.A. Assad [1] there is $(x_n) \in X^{\mathbb{N}}$ such that $x_n = (1 - \frac{1}{n+1})f(x_n)$ for $n \in \mathbb{N}$. The boundedness of (x_n) follows from Lemma 2.23; Lemma 2.7 completes the proof.

Remark 2.25 (i) Theorem 2.24 was originally proved for Hilbert spaces by J. Reiner mann and R. Schöneberg [16].

(ii) In the case of a Hilbert space $(E, (\cdot, \cdot))$ and a convex X Theorem 2.24 remains valid if condition (i) is cancelled.

	$\begin{matrix} h(\cos \varphi, \sin \varphi, x) \\ \varphi \in \mathbb{R}^+, x \in [0, \frac{1}{2}] \end{matrix}$	$\begin{matrix} h(\tau^{-1}(\varphi) \cos \varphi, \tau^{-1}(\varphi) \sin \varphi, x) \\ 0 \leq \varphi \leq \pi \end{matrix}$	$\begin{matrix} h(\tau^{-1}(\varphi) \cos \varphi, \tau^{-1}(\varphi) \sin \varphi, x), \varphi \geq \pi \\ h(x \cos \varphi, x \sin \varphi, 1), \frac{2}{3} \leq x < 1 \end{matrix}$	$\begin{matrix} h(\cos \varphi, \sin \varphi, x) \\ \varphi \in \mathbb{R}^+ \end{matrix}$
$x \in [3/2, 2]$	/	$(\tau^{-1}(\varphi + \pi)(x-1)) \cos((\varphi + \pi)(x-1)), \dots$ $\dots, 2 - \frac{\varphi}{\pi} (1 - \frac{x}{2})$	$(\tau^{-1}(\varphi - 3\pi + 2\pi x) \cos(\varphi - 3\pi + 2\pi x), \dots$ $\dots, 1 + \frac{x}{2})$	$(\cos(\varphi - 3\pi + 2\pi x), \dots$ $\dots, 1 + \frac{x}{2})$
$x \in [1, 3/2]$	/	$(\tau^{-1}(\varphi + \pi)(x-1)) \cos((\varphi + \pi)(x-1)), \dots$ $\dots, 2 - \frac{\varphi}{\pi} (\frac{5}{2} - \frac{3}{2}x)$	$(\tau^{-1}(\varphi - 3\pi + 2\pi x) \cos(\varphi - 3\pi + 2\pi x), \dots$ $\dots, \frac{3}{2}x - \frac{1}{2})$	$(\cos(\varphi - 3\pi + 2\pi x), \dots$ $\dots, \frac{3}{2}x - \frac{1}{2})$
$x = 1$	$(\frac{1}{2}, 0, 2)$	● see below	$(\tau^{-1}(\tau(x) - \pi) \cos(\varphi - \pi), \dots, 1)$	$(\cos(\varphi - \pi), \sin(\varphi - \pi), 1)$

● $\frac{1}{2} \leq x \leq \frac{2}{3}$, $\varphi \in [-\pi, \pi]$, $m(\varphi) := \tau^{-1}(|\varphi|) + \frac{1}{2} (\frac{2}{3} - \tau^{-1}(|\varphi|))$:

	$\frac{1}{2} \leq x \leq \tau^{-1}(\varphi)$	$\tau^{-1}(\varphi) \leq x \leq m(\varphi)$	$m(\varphi) \leq x \leq \frac{2}{3}$
$\varphi \neq 0$ $ \varphi \neq \pi$	$(\frac{1}{2}, 0, 2 - \frac{(x - \frac{1}{2}) \cdot \varphi }{(\tau^{-1}(\varphi) - \frac{1}{2}) \cdot \pi})$	$(\frac{1}{2}, 0, 2 - \frac{ \varphi }{\pi} - \frac{(x - \tau^{-1}(\varphi))}{m(\varphi) - \tau^{-1}(\varphi)} (1 - \frac{ \varphi }{\pi}))$	$(\frac{1}{2} \cos((\frac{x - m(\varphi)}{\frac{2}{3} - m(\varphi)})(\varphi - \pi)), \frac{1}{2} \sin(\dots), 1)$
$\varphi = 0$	/	$(\frac{1}{2}, 0, 2 - 6(2x - 1))$	$(\frac{1}{2} \cos((\varphi - 12\pi)x), \frac{1}{2} \sin(\dots), 1)$
$ \varphi = \pi$	$(\frac{1}{2}, 0, 2 - 6(x - \frac{1}{2}))$	/	/

R e f e r e n c e s

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