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Václav Koubek

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GRAPHS WITH GIVEN SUBGRAPHS REPRESENT ALL CATEGORIES

Václav KOUBEK, Praha

Abstract: Let G be an arbitrary finite graph without loops. Denote by GRA_G a full subcategory of the category of all graphs and compatible mappings generated by all graphs such that for each edge there exists their full subgraph isomorphic to G containing this edge. We prove that there exists a strong embedding the category of all graphs into GRA_G, in particular, GRA_G is binding.

Key words: Full subcategory, binding category, graphs with given subgraphs.

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It is well-known that for every monoid M there exists a graph (X,R) such that the endomorphism monoid of (X,R) is isomorphic to M, and, if M is finite then X can be finite, too. Z. Hedrlín and L. Kučera obtained a stronger result: every concrete category can be fully embedded into the category GRA of all graphs. This has lead to the next important question: Into which categories the category GRA can be fully embedded? When solving this problem we often see that it is much easier to embed into a given category not directly GRA but rather another category, into which GRA can be embedded. To this end, we use some full subcategories of GRA, e.g. the category of all undirected graphs, of all connected graphs etc. Therefore we have to know which full subcategories of the category GRA are binding (i.e. the category GRA can be fully embedded into them). For instance, this question was solved in the following papers [2,4,5,6,8,9,10,11].

The aim of this note is to prove that for every finite graph (X,R) without loops such that $R \neq \emptyset$ there exists a strong embedding of the category GRA to its full subcategory which contains those graphs, each edge of which lies in a full subgraph, isomorphic to (X,R).

<u>Definition</u> [12]. Let (K,U), (L,V) be concrete categories. A full embedding $\Phi : K \longrightarrow L$ is called a strong embedding if there exists a set functor F: Set $\longrightarrow$ Set such that the following diagram commutes

$$\begin{array}{ccc} K & \xrightarrow{\ \Phi\ } & L \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle V} \\ \mathrm{Set} & \xrightarrow[\ F\ ]{} & \mathrm{Set} \end{array}$$

We use a modification of a general construction of E. Mendelsohn [10]. We shall define a šíp-product (or šíp-součin) $(X,R,R',A,B) * (Y,S)$ of a šíp (X,R,R',A,B) and an arbitrary graph (Y,S) where X is a set, $R' \subset R \subset X \times X$, i.e. R, R' are relations on X, A, B are disjoint subsets of X such that there exists a bijection $i: A \longrightarrow B$, $i \times i(R \cap (A \times A)) = R \cap (B \times B)$ and $i \times i(R' \cap (A \times A)) = R' \cap (B \times B)$. Now, $(X,R,R',A,B) * (Y,S)$ is a quotient graph of $(X \times Y \times Y, T = \{((x_1,y_1,y_2),(x_2,y_1,y_2)); (x_1,x_2) \in R, (y_1,y_2) \in S\} \cup \{((x_1,y_1,y_2), (x_2,y_1,y_2)); (x_1,x_2) \in R', (y_1,y_2) \in ((Y \times Y) - S)\})$ under the equivalence $\sim$ which is defined as follows: $(x_1,y_1,u_1) \sim (x_2,y_2,u_2)$ whenever

either $x_1 = x_2 \in A$ and $y_1 = y_2$

or $i(x_1) = x_2$ and $y_1 = u_2$

or $x_1 = x_2 \in B$ and $u_1 = u_2$.

Intuitively, the šíp-product is obtained by replacing every arrow of the graph (Y,S) with the starting point a and the endpoint b of a copy of the graph (X,R), where the set A replaces the point a and B replaces b and every arrow of $(Y \times Y) - S$ with the starting point a and the endpoint b by a copy of the graph (X,R') where A replaces a and B replaces b.

Let $f: (Y,S) \longrightarrow (Y',S')$ be a compatible mapping, then a mapping $f^*: (X,R,R',A,B) * (Y,S) \longrightarrow (X,R,R',A,B) * (Y',S')$ defined by $f^*(x,y_1,y_2) = (x,f(y_1),f(y_2))$ is compatible and therefore $\Phi(Y,S) = (X,R,R',A,B) * (Y,S)$, $\Phi f = f^*$ is a functor. Notice that $f^* = ((C_A \times I) \vee (Q_2 \times C_{X-(A\cup B)}))f$ where C_A or $C_{X-(A\cup B)}$ are constant set functors to A or $X - (A \cup B)$, I is the identity set functor and Q_2 is the set hom-functor to two-point set. Hence, if Φ is a full embedding then it is a strong embedding.

<u>Definition</u>. A šíp (X,R,R',A,B) is called strongly rigid if for every graph (Y,S) and every compatible mapping $f: (X,R) \longrightarrow (X,R,R',A,B) * (Y,S)$ (or $f: (X,R') \longrightarrow (X,R,R',A,B) * (Y,S)$) there exists $(y_1,y_2) \in S$ (or $(y_1,y_2) \in Y \times Y$) with $f(x) = [(x,y_1,y_2)]$ for every $x \in X$ ($[(x,y_1,y_2)]$ is the class of $\sim$ containing (x,y_1,y_2)).

<u>Proposition 1</u>. If (X,R,R',A,B) is strongly rigid then Φ is a strong embedding.

Proof. It suffices to prove that Φ is full. The proof is an easy modification of the proof in [10]. Let $(Y,S),(Y',S')$ be graphs and let $f: (X,R,R',A,B) * (Y,S) \longrightarrow (X,R,R',A,B) * * (Y',S')$ be a compatible mapping. Since (X,R,R',A,B) is strongly rigid we get that for every couple $(y_1,y_2) \in Y \times Y$ there exists $(u_1,u_2) \in Y' \times Y'$ with $f([(x,y_1,y_2)]) = [(x,u_1,u_2)]$ for every $x \in X$ and, moreover, if $(y_1,y_2) \in S$ then $(u_1,u_2) \in S'$. Therefore, we can define $h: Y \times Y \longrightarrow Y' \times Y'$ by $h(y_1,y_2) = (u_1,u_2)$. Further if $y_1,y_2,y_3 \in Y$, then $f([(x,y_1,y_2)]) = f([(x,y_1,y_3)])$ for every $x \in A$ and so if $h(y_1,y_2) = (u_1,u_2)$, $h(y_1,y_3) = (u_3,u_4)$ then $u_1 = u_3$. Analogously, we prove that if $h(y_1,y_2) = (u_1,u_2)$ and $h(y_3,y_2) = (u_3,u_4)$ then $u_2 = u_4$. Therefore there exist $g_1,g_2: Y \longrightarrow Y'$, with $h = g_1 \times g_2$. Further $f([(x_1,y_1,y_2)]) = f([(x_2,y_3,y_1)])$ whenever $x_1 \in A$ and $i(x_1) = x_2$, hence $g_1(y_1) = g_2(y_1)$ and thus $g_1 = g_2$. Therefore $h = g \times g$ (where $g_1 = g = g_2$) and because $h(S) \subset S'$ we get that g is compatible. Clearly $g^* = f$.

We shall construct a šíp with special properties and therefore we shall need special rigid graphs (i.e. graphs which have no non-identical endomorphism).

Definition. Let (X,R) be a graph, $x,y \in X$. A sequence $\{K_i\}_{i=1}^{m}$, $K_i \subset X$ such that card $K_i = n$, card $(K_i \cap K_{i+1}) = n - 1$, $(K_i, R \cap (K_i \times K_i))$ is a complete graph without loops for every $i = 1,2,\dots,m$ is an n-path connecting x with y in (X,R) if $x \in K_1$, $y \in K_m$.

Note. If $f: (X,R) \longrightarrow (Y,S)$ is a compatible mapping and (Y,S) has not loops t... maps every n-path into an n-path.

Lemma 2. For every triple (m,n,p) of natural numbers such that m is a non-trivial multiple of n, $n > p + 2$ there exists a graph $I^m_{n,p} = (M_m, Q_{n,p})$ where $M_m = \{0,1,2,\dots,m\}$ such that

1) for every distinct points $x,y \in M_m$ there exists an n-path connecting x with y;

2) for every edge $(x,y) \in Q_{n,p}$ there exists $Z \subset M_m$ such that $x,y \in Z$, card $Z \geq p$ and $(Z, Q_{n,p} \cap (Z \times Z))$ is a complete graph without loops (i.e. $I^m_{n,p}$ has not loops and it is symmetric);

3) there exists an edge $(x,y) \in Q_{n,p}$ with the following property: for every $Z \subset M_m$ such that $x,y \in Z$ and $(Z, Q_{n,p} \cap \cap (Z \times Z))$ is a complete graph without loops, card $Z \leq p$;

4) the chromatic number of $(Z, Q_{n,p} \cap (Z \times Z))$ is $n + 1$ iff $Z = M_m$;

5) $I^m_{n,p}$ is rigid;

6) if $f: I^m_{n,p} \longrightarrow I^{m'}_{n,p'}$ is compatible then $m \geq m'$ and $p \leq p'$, moreover, if $m = m'$ then f is compatible iff $p \leq p'$ and f is the identity mapping;

7) for every $x \in M_m$, card $\{y;\ (x,y) \in Q_{n,p}\} \leq 2n$.

Proof see [9].

Definition. For a triple (m,n,p) of natural numbers such that m is a non-trivial multiple of n, $n > p + 2$ define

$P_{n,p} = \{(x,y);\ x < y, (x,y) \in Q_{n,p}\}$.

Clearly, $Q_{n,p} = \{(x,y);\ (x,y) \in P_{n,p}$ or $(y,x) \in P_{n,p}\}$.

Construction 3. Let $G_0 = (X_0,R_0)$ be a connected graph without loops such that $\aleph_0 > \mathrm{card}\, X_0 > 1$. Then for arbitrary natural numbers n_0,p_0 such that $p_0 > \mathrm{card}\, X_0$, $n_0 > p_0 + (4 \cdot \mathrm{card}\, X_0) - 6$ we construct a šíp $\mathcal{S}(G_0,n_0,p_0) = (Z,T,T',A,B)$. First assume that $\mathrm{card}\, X_0 > 2$. Choose $(x_0,y_0) \in R_0$. Choose a bijection $\varphi : \{0,1,\dots,\mathrm{card}\, X_0 - 3\} \longrightarrow X_0 - \{x_0,y_0\}$ and identify i with $\varphi(i)$, then $X_0 = \{0,1,\dots,\mathrm{card}\, X_0 - 3,\ x_0,y_0\}$.

For $i = 0,1,\dots,(2 \cdot \mathrm{card}\, X_0) - 5$ denote by $m_i = m_{i+(2\cdot \mathrm{card}\, X_0)-4} = n_0 \cdot (p_0 + i) \cdot (p_0 + i + (2 \cdot \mathrm{card}\, X_0) - 4)$. Put

$$Z = \bigcup_{i=0}^{(2 \cdot \mathrm{card}\, X_0) - 5} M_{m_i} \times \{i\}$$

We shall define $T_1,T_2,T_3,T_4,T_5,T_6 \subset Z \times Z$.

For every $j = 0,1,\dots,2n_0$, choose $x_j^i \in M_i$ where $i = 0,1,\dots,(4 \cdot \mathrm{card}\, X_0) - 9$. Further, for every $i = 0,1,\dots,(4 \cdot \mathrm{card}\, X_0) - 9$, by Condition 7 in Lemma 2 there exists a decomposition $\{W_j^i;\ j = 0,1,\dots,2n_0\}$ of P_{n_0,p_0+i} such that if $(x,y),(z,v) \in W_j^i$ then $x \neq v$, $y \neq z$ and, moreover, $x \neq x_j^i \neq y$ (of course $z \neq x_j^i \neq v$, too).

Now, if $(k_1,k_2) \in R_0$ then

$((x_j^{2k_1},2k_1),(x_j^{2k_2},2k_2))$, $((x_j^{2k_1+1},2k_1 + 1),\ (x_j^{2k_2+1},2k_2 + 1)) \in T_1 \cap T_2$ for every $j = 0,1,\dots,2n_0$;

if $(k_1,x_0),(x_0,k_2),(k_3,y_0),(y_0,k_4) \in R_0$ then

$((x_j^{2k_1},2k_1),(u,i)),((u,i),(x_j^{2k_2},2k_2)) \in T_1$ if i is odd and there exists v with $(u,v) \in W_j^i$,

$((x_j^{2k_1+1},2k_1 + 1),(u,i)),((u,i),(x_j^{2k_2+1},2k_2 + 1)) \in T_1$ if i is even and there exists v with $(u,v) \in W_j^i$,

$((x_j^{2k_3}, 2k_3), (v,i)), ((v,i), (x_j^{2k_4}, 2k_4)) \in T_1$ if i is odd and there exists u with $(u,v) \in W_j^i$,

$((x_j^{2k_3+1}, 2k_3 + 1), (v,i)), ((v,i), (x_j^{2k_4+1}, 2k_4 + 1)) \in T_1$ if i is even and there exists u with $(u,v) \in W_j^i$,

$((x_j^{2k_1}, 2k_1), (u,i)), ((u,i), (x_j^{2k_2}, 2k_2)) \in T_2$ if i is odd and there exists v with $(u,v) \in W_j^{i-2+2\cdot \text{card } X_0}$,

$((x_j^{2k_1+1}, 2k_1 + 1), (u,i)), ((u,i), (x_j^{2k_2+1}, 2k_2 + 1)) \in T_2$ if i is even and there exists v with $(u,v) \in W_j^{i-2+2\cdot \text{card } X_0}$,

$((x_j^{2k_3}, 2k_3), (v,i)), ((v,i), (x_j^{2k_4}, 2k_4)) \in T_2$ if i is odd and there exists u with $(u,v) \in W_j^{i-2+2\cdot \text{card } X_0}$,

$((x_j^{2k_3+1}, 2k_3 + 1), (v,i)), ((v,i), (x_j^{2k_4+1}, 2k_4 + 1)) \in T_2$ if i is even and there exists u with $(u,v) \in W_j^{i-2+2\cdot \text{card } X_0}$,

if $(u,v) \in P_{n_0, p_0+i}$ and $i \leq (2 \cdot \text{card } X_0) - 5$ then

$((u,i), (v,i)) \in T_3$

$((u,i), (v,i)), ((v,i), (u,i)) \in T_4$

if $(u,v) \in P_{n_0, p_0+i}$ and $i > (2 \cdot \text{card } X_0) - 5$ then

$((u, i + 2 - (2 \cdot \text{card } X_0)), (v, i + 2 - (2 \cdot \text{card } X_0))) \in T_5$

$$\left.\begin{array}{l} ((u, i + 2 - (2 \cdot \text{card } X_0)), (v, i + 2 - (2 \cdot \text{card } X_0))) \\ ((v, i + 2 - (2 \cdot \text{card } X_0)), (u, i + 2 - (2 \cdot \text{card } X_0))) \end{array}\right\} \in T_6$$

Put $T = T_2 \cup T_5$, $T' = T_1 \cup T_3$ if $(y_0, x_0) \notin R_0$, $T = T_2 \cup T_6$, $T' = T_1 \cup T_4$ if $(y_0, x_0) \in R_0$. Further choose distinct points $a, b \in Z$ and put $A = \{a\}$, $B = \{b\}$. Since id: $I_{n_0, p_0+i}^{m_i} \longrightarrow I_{n_0, p_0+i-2+(2\cdot \text{card } X_0)}^{m_i}$ is compatible, we get that $T' \subset T$.

If card $X_0 = 2$ then $\mathcal{G}(G_0,n_0,p_0)$ is constructed for $p_0 > 2$, $n_0 > p_0 + 3$ and we put $Z = M_{n_0 p_0}$, $T = P_{n_0,p_0+1}$, $T' = P_{n_0,p_0}$ if G_0 is not symmetric, $T = Q_{n_0,p_0+1}$, $T' = Q_{n_0,p_0}$ if G_0 is symmetric. Choose distinct points $a,b \in Z$ and put $A = \{a\}$, $B = \{b\}$. Clearly $T' \subset T$.

<u>Lemma 4</u>. Let $G_0 = (X_0,R_0)$ be a connected graph without loops such that $\aleph_0 > \text{card } X_0 > 1$. Then for every edge $(x,y) \in T$ or $(x,y) \in T'$ of $\mathcal{G}(G_0,n_0,p_0)$ there exists a full subgraph of $\mathcal{G}(G_0,n_0,p_0)$ isomorphic to G_0 and containing (x,y).

Proof. Put $\mathcal{G}(G_0,n_0,p_0) = (Z,T,T',A,B)$ where $T' = T_1 \cup T_3$ (or $T_1 \cup T_5$) and $T = T_2 \cup T_4$ (or $T_2 \cup T_6$). If $(x,y) \in T_3 \cup T_4 \cup T_5 \cup T_6$ then there exists i such that $x = (u,i)$, $y = (v,i)$ and

1) $(u,v) \in P_{n_0,p_0+i}$ if $(x,y) \in T_3$,

2) $(u,v) \in Q_{n_0,p_0+i}$ if $(x,y) \in T_4$,

3) $(u,v) \in P_{n_0,p_0+i-2+(2\cdot \text{card } X_0)}$ if $(x,y) \in T_5$,

4) $(u,v) \in Q_{n_0,p_0+i-2+(2\cdot \text{card } X_0)}$ if $(x,y) \in T_6$.

Then there exists $j \in \{0,1,\dots,2n_0\}$ such that

a) $(u,v) \in W_j^i$ or $(v,u) \in W_j^i$ if $(x,y) \in T_3 \cup T_4$,

b) $(u,v) \in W_j^{i-2+(2\cdot \text{card } X_0)}$ or $(v,u) \in W_j^{i-2+(2\cdot \text{card } X_0)}$ if $(x,y) \in T_5 \cup T_6$.

Put $Z' = \{(x_j^k,k);\ k+1 \text{ is odd}\} \cup \{x,y\}$. Then $(Z', T \cap (Z' \times Z'))$ or $(Z', T' \cap (Z' \times Z'))$ is isomorphic to G_0.

If $(x,y) \in T_1 \cup T_2$ then there exist $i \in \{0,1,\dots,\text{card } X_0 - 3\}$, $j \in \{0,1,\dots,2n_0\}$ with $(x_j^{2i},2i) \in \{x,y\}$ or $(x_j^{2i+1},2i+1) \in \{x,y\}$; assume that $x = x_j^{2i}$ (the proof for the other case is analogous). If $y = (x_j^{2k},2k)$ for some $k \in \{0,1,\dots,\text{card } X_0 - 3\}$

then choose $i' \in \{0,1\}$ such that $i + i'$ is odd and (u,i'), (v,i') with $(u,v) \in W_j^i$, $((u,i'),(v,i')) \in T$ if $(x,y) \in T_2$, $((u,i'),(v,i')) \in T'$ if $(x,y) \in T_1$. Put $Z' = \{(x_j^k; k); k + i$ is even$\} \cup \{(u,i'),(v,i')\}$. It is clear that a full subgraph on Z' is isomorphic to G_0.
If $y = (u,i')$ then $i + i'$ is odd. Choose (v,i') such that $(u,v) \in W_j^i$ and $((u,i'),(v,i')) \in T$ if $(x,y) \in T_2$, $((u,i'),(v,i')) \in T'$ if $(x,y) \in T_1$. Put $Z' = \{(x_j^k,k); k + i$ is even$\} \cup \cup \{(u,i'),(v,i')\}$ and, again, the full subgraph on Z' is isomorphic to G_0.

Proposition 5. For every connected graph $G_0 = (X_0,R_0)$ without loops where $\aleph_0 > \text{card } X_0 > 1$, the sip $\mathcal{G}(G_0,n_0,p_0)$ is strongly rigid.

Proof. Let $\mathcal{G}(G_0,n_0,p_0) = (Z,T,T',A,B)$ and let (Y,S) be an arbitrary graph. Assume that $f\colon (Z,T) \to \mathcal{G}(G_0,n_0,p_0) * (Y,S)$ is a compatible mapping. Put $T^* = \{(x,y); (x,y) \in T$ or $(y,x) \in \in T\}$. Denote by $\mathcal{G}(G_0,n_0,p_0) * (Y,S) = (Y^*,S')$ and put $S^* = = \{(x,y); (x,y) \in S'$ or $(y,x) \in S'\}$. Then $f\colon (Z,T^*) \longrightarrow (Y^*,S^*)$ is a compatible mapping. Since (Y^*,S^*) has not loops, we see that f preserves n_0-paths. Hence, by Lemma 2 for every $i = = 0,1,\ldots,2 \cdot \text{card } X_0 - 5$ there exist $y_i,z_i \in Y$ with $f(x,i) = = [(x,i,y_i,z_i)]$ for every $x \in M_{m_i}$. Further the restriction T^* to $M_{m_i} \times \{i\}$ is isomorphic to $I^{m_i}_{n_0,p_0+i-4+(2\cdot \text{card } X_0)}$ and thus $(y_i,z_i) \in S$. We are to prove that if $f(x,i_0) = [(x,i_0,y_{i_0},z_{i_0})]$ and $f(x,i_1) = [(x,i_1,y_{i_1},z_{i_1})]$ where $i_0,i_1 = 0,1,\ldots$ $\ldots,(2\cdot \text{card } X_0) - 5$ then $y_{i_0} = y_{i_1}$ and $z_{i_0} = z_{i_1}$. It follows from the fact that there exist distinct points x_1, x_2, x_3, x_4 with $((x_1,i_0),(x_2,i_1)),((x_3,i_0),(x_4,i_1)\in T^*$ and if $((x_1,i_0,y_{i_0},z_{i_0}),\ (x_2,i_1,y_{i_1},z_{i_1})),((x_3,i_0,y_{i_0},z_{i_0}),(x_4,i_1,y_{i_1},z_{i_1}))\in S^*$ where $(y_{i_0},z_{i_0})\neq(y_{i_1},z_{i_1})$ then either $x_1 = x_3$ or $x_2 = x_4$ - a contradiction. Hence there exists $(y_1,y_2)\in S$ with $f(z) = [(z,y_1,y_2)]$ for every $z\in Z$. If $f\colon (Z,T') \longrightarrow$ $\longrightarrow \varphi(G_0,n_0,p_0) * (Y,S)$ then the proof is analogous.

<u>Definition</u>. Let $G = (X,R)$ be a graph. Denote by GRA_G the full subcategory of GRA consisting of those graphs (Y,S) which fulfil: for every edge $(x,y)\in S$ there exists $Z\subset Y$ with $x,y\in Z$ such that $(Z,S\cap(Z\times Z))$ is isomorphic to G.

<u>Main Theorem 6</u>. Let $G = (X,R)$ be a finite non-trivial graph without loops. Then there exists a strong embedding from GRA into GRA_G.

Proof. Let $G_1 = (X_1,R_1)$, $G_2 = (X_2,R_2)\ldots G_m = (X_m,R_m)$ denote all components of G with $R_i \neq \emptyset$. Choose a sequence $p_1, p_2,\ldots$ $\ldots, p_m$ with card $X < p_1 < p_2 < \ldots < p_m$ and a sequence of $n_1, n_2,\ldots$ $\ldots, n_m$ with $n_i \cdot p_i \cdot (2\cdot \text{card } X_i - 4 + p_i) > \text{card } Z_{i-1}$ where $\varphi(G_i,n_i,p_i) = (Z_i,T_i,\ T_i',A_i,B_i)$ for every $i = 1,2,\ldots,m$ and $n_1 > p_m - 6 + 4\cdot \text{card } X$. Define $\psi\colon GRA \longrightarrow GRA_G$ as follows:

$$\psi(Y,S) = \varphi(G_1,n_1,p_1) * (Y,S) \vee (\bigvee_{i=2}^{m} (Z_i,T_i))$$

where $\vee$ denotes the disjoint union and for $i = 1,2,\ldots,m$ $\varphi(G_i,n_i,p_i) = (Z_i,T_i,T_i',A_i,B_i)$. For $i = 1,2,\ldots,m$ define:

ψf on (Z_i,T_i) is the identity mapping;

further, ψf on $\varphi(G_1,n_1,p_1) * (Y,S)$ is Φf where Φ is the embedding induced by $\varphi(G_1,n_1,p_1)$. Since $p_1 >$ card X we get

that $\psi(Y,S) \in GRA_G$, hence $\psi : GRA \longrightarrow GRA_G$.
Further, clearly, ψ is an embedding and if U is a forgetful functor from GRA to Set then there exists a set functor F: Set $\longrightarrow$ Set with $F \circ U = U \circ \psi$ (because Φ is a strong embedding by Propositions 1 and 5). Since either (Z_1,T_1) or (Z_1,T_1') is isomorphic to some full subgraph of $\mathcal{C}(G_1,n_1,p_1)*$ $*(Y,S)$, it suffices to prove that if $f: (Z_i,S_i) \longrightarrow (Z_j,S_j)$ is compatible then $i = j$ and f is the identity mapping where $S_i = T_i$ or $= T_i'$ and $S_j = T_j$ or T_j', $i,j = 1,2,\dots,m$. Denote $S_i^* = \{(x,y);\ (x,y) \in S_i \text{ or } (y,x) \in S_i\}$ and $S_j^* = \{(x,y);$ $(x,y) \in S_j \text{ or } (y,x) \in S_j\}$.
Since (Z_j,S_j^*) has no loop we get if $x,y \in Z$ are connecting with 5-path in (Z_i,S_i^*) then $f(x), f(y)$ are connecting with 5-path in (Z_j,S_j^*), too. Therefore by the choice of n_i and p_i and by Condition 6 in Lemma 2 we obtain that $i = j$.
Since $\mathcal{G}(G_i,n_i,p_i) * (\{x,y\},\{(x,y)\}) = (Z_i,T_i)$ and $\mathcal{G}(G_i,$ $n_i,p_i) * (\{x,y\},\emptyset) = (Z_i,T_i')$ we get by Proposition 5 that f is the identity mapping. The proof is concluded.

<u>Corollary 7</u>. For a finite graph G the category GRA_G is binding iff G has not loops and has at least one edge.

<u>Corollary 8</u>. In the finite set theory GRA_G is binding iff G has not loops and has at least one edge.

Proof follows from the fact that $\mathcal{G}(G,n,p)$ is finite for every graph G and every couple (n,p) of natural numbers.

<u>Corollary 9</u>. For every finite graph G without loops with at least one edge and for every (finite) monoid M there exist infinitely many (finite) graphs (Y,S) such that:

1) for every edge $(x,y) \in S$ there exists $Z \subset Y$ such that $x,y \in Z$ and $(Z, S \cap (Z \times Z))$ is isomorphic to G;

2) the endomorphism monoid of (Y,S) is isomorphic to M;

3) there exists no compatible mapping between them.

Moreover, there exist strong embeddings ψ_i: GRA $\longrightarrow$ $\longrightarrow$ GRA_G, $i = 1,2,\ldots$ such that for every couple of graphs (Y,S), (Y',S') and for every $i \neq j$ there exists no compatible mapping f: $\psi_i(Y,S) \longrightarrow \psi_j(Y',S')$.

Proof. This assertion is obtained by a suitable choice of n, p, by Lemma 2 (Condition 6).

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Matematicko-fyzikální fakulta

Karlova universita

Malostranské nám. 25, 11800 Praha 1

Československo

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