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ON COMPACT SPACES WHICH ARE UNIONS OF CERTAIN COLLECTIONS OF SUBSPACES OF SPECIAL TYPE, II.

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Abstract: This article is a natural continuation to the previous one, published under the same title. One of the main results is the following theorem: if X is a compact and $X = \cup\{X_i : i = 1,\dots,k\}$ where each X_i is a space with uniform base then X is sequential and, for each $A \subset X$, $c\ell(A) = \operatorname{seqc}\ell_k(A)$, where $\operatorname{seqc}\ell_0(A) = A$ and $\operatorname{seqc}\ell_k(A) = \operatorname{seqc}(\operatorname{seqc}\ell_{k-1}(A))$ for $k \ge 1$. Another result: if X is a compact which is the union of $\le \aleph_1$ of metrizable subspaces then either X is finite or there exists a non-trivial converging sequence in X. We also formulate some open problems and describe an example.

Key words and expressions: Sequential space, tightness, uniform base, network.

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0. Notations and conventions. We use the same notations and conventions as in the previous article [1]. In addition, for $k \in N = \{0,1,2,\dots\}$ and $A \subset X$, where X is a space, we define: $\operatorname{seqc}\ell_0(A) = A$ and $\operatorname{seqc}\ell_{k+1}(A) = \operatorname{seqc}\ell(\operatorname{seqc}\ell_k(A)$.

1. Theorem. Let $k \in N^+ = \{1,2,\dots\}$, $k \ge 2$, and $X = \cup\{X_i : i = 1,\dots,k\}$, where X is an $\aleph_0$-compact and, for each $i = 1,\dots,k$, X_i satisfies the following conditions:
a) if $A \subset X_i$ and $|A| \le \aleph_0$ then the closure of A in X_i is

a Fréchet-Urysohn space with a countable network; b) $\psi(X_i) \leq \aleph_0$, and c) $X_i \in \mathcal{C}_{\aleph_0}$. Then the space X is sequential and, for each $A \subset X$, $c\ell(A) = \mathrm{seqc}\ell_{k-1}(A)$.

Proof. For $k = 2$ our assertion coincides with Theorem II.8 in [1]. Assume that it is true for all $k \in N^+$ such that $2 \leq k \leq r$, for some $r \in N^+$, $r \geq 2$, and let us prove the theorem for $k = r + 1$. Let $A \subset X$ and $x \in c\ell(A)$. By Theorem I.12 of [1], $t(X) \leq \aleph_0$. Thus there exists $A^* \subset A$ such that $|A^*| \leq \aleph_0$ and $x \in c\ell(A^*)$. Put $A_i^* = A^* \cap X_i$. Clearly, $x \in c\ell(A_{i^*}^*)$ for some $i^* \in \{1,\dots,r+1\}$. The space $X^* = c\ell(A_{i^*}^*)$ is an $\aleph_0$-compact. If $x \in X_{i^*}$ then a) implies that $x \in \mathrm{seqc}\ell(A_{i^*}^*)$. Then it follows that $x \in \mathrm{seqc}\ell(A) \subset \mathrm{seqc}\ell_{r+1}(A)$ and the argument in this case is complete.

Let $x \notin X_{i^*}$. We fix $\tilde{i} \in \{1,\dots, r+1\}$ for which $x \in X_{\tilde{i}}$. Then $i^* \neq \tilde{i}$. Let us fix a countable network γ^* in the closure B_{i^*} of the set $A_{i^*}^*$ in the space X_{i^*} such that $x \notin c\ell(P)$ for every $P \in \gamma^*$. As $x \notin B_{i^*}$, this is possible in view of a). We also fix a countable family $\tilde{\gamma}$ of open sets in X^* such that $(\bigcap\tilde{\gamma}) \cap X_{\tilde{i}} = \{x\}$. This is possible in view of b). Put $G = (\bigcap\tilde{\gamma}) \cap (\bigcap \{X^* \setminus c\ell(P) : P \in \gamma^*\})$. Clearly G is a G_δ-set in X^* and $x \in G$. Hence there exists a closed set F in X^* such that $x \in F \subset G$ and $\chi(F, X^*) \leq \aleph_0$ (see [2],[3]). Let us fix a base $\mathcal{B} = \{\mathcal{U}_n : n \in N^+\}$ of F in X^* such that $\mathcal{U}_{n+1} \subset \mathcal{U}_n$ for all $n \in N^+$. Observe that $B_{i^*} \cap (\bigcap \{X^* \setminus c\ell(P) : P \in \gamma^*\}) = \emptyset$ and $X_{i^*} \cap X^* = B_{i^*}$. Hence $(\bigcap \{X^* \setminus c\ell(P) : P \in \gamma^*\}) \cap X_{i^*} = \emptyset$. It follows that $G \cap X_{i^*} = \emptyset$ and $F \cap X_{i^*} = \emptyset$. As $F \subset \bigcap\tilde{\gamma}$, we have: $F \cap X_{\tilde{i}} = \{x\}$. We claim that $x \in c\ell(S)$ where $S = F \cap \mathrm{seqc}\ell(A_{i^*}^*)$. Assume the contrary and fix a neighborhood V of x in X such that $c\ell(V) \cap c\ell(S) = \emptyset$. For every $n \in N^+$ the set $\mathcal{U}_n \cap V$ is open in X^* and $x \in \mathcal{U}_n \cap V$. From $x \in c\ell(A^*_{i*})$ it follows that $\mathcal{U}_n \cap V \cap A^*_{i*} \neq \emptyset$. We choose $x_n \in \mathcal{U}_n \cap V \cap A^*_{i*}$ and put $C = \{x_n : n \in N^+\}$, $C^* = c\ell(C) \setminus C$. Clearly $F \supset c\ell(C) \setminus C \neq \emptyset$. As $C \subset V$ and $c\ell(V) \cap c\ell(S) = \emptyset$, we have: $c\ell(C) \cap c\ell(S) = \emptyset$. Thus the space $c\ell(C)$ is not sequential - otherwise at least one point of the set $c\ell(C) \setminus C$ would be contained in S. It follows that $|c\ell(C)| > \aleph_0$. Hence $C^* \neq \{x\}$. We fix $z \in C^* \setminus \{x\}$ and a neighborhood W of z in $c\ell(C)$ such that $c\ell(W) \not\ni x$. Then $C' = C \cap W$ is an infinite set and $F' = c\ell(C') \subset X_{i*} \cup (F \setminus \{x\}) \subset \bigcup\{X_i : i \in \{1,\dots$ $\dots, r+1\} \setminus \{\tilde{i}\}\}$ (as $(F \setminus \{x\}) \cap X_{\tilde{i}} = \emptyset$). Thus $F' =$ $= \bigcup\{X_i \cap F' : i \in \{1,\dots,r+1\} \setminus \{\tilde{i}\}\}$, where $X_i \cap F'$ satisfy the conditions a), b) and c) and F is an $\aleph_0$-compact. By the inductive hypothesis, F' is sequential. The set C' is not closed in F' as $z \in c\ell(C') \setminus C'$. Hence $z' \in \mathrm{seq}c\ell(C')$ for some $z' \in c\ell(C') \setminus C'$. As C is discrete, $z' \in c\ell(C) \setminus C \subset F$. It follows that $z' \in c\ell(A^*_{i*}) \cap F \subset S$ - in contradiction with $z' \in$ $\in c\ell(C') \subset c\ell(C) \subset c\ell(V) \subset X^* \setminus c\ell(S)$. We have proved by the argument that $x \in c\ell(S)$.

From $F \cap X_{i*} = \emptyset$ it follows that $F = \bigcup\{X_i \cap F : i \in \{1,\dots$ $\dots, r+1\} \setminus \{i^*\}\}$. As $S \subset F$, by the inductive hypothesis we have: $c\ell(S) = \mathrm{seq}c\ell_{r-1}(S)$. But $S \subset \mathrm{seq}c\ell(A^*_{i*})$. Thus $\mathrm{seq}c\ell_r(A^*_{i*}) \supset \mathrm{seq}c\ell_{r-1}(S) = c\ell(S) \ni x$. Hence $\mathrm{seq}c\ell_r(A) \ni x$. As x is any point of the set $c\ell(A)$ we finally obtain: $\mathrm{seq}c\ell_r(A) = c\ell(A)$. Theorem 1 is proved.

2. <u>Corollary</u>. If X is an $\aleph_0$-compact and X is the union of a finite collection of spaces with uniform bases then

X is sequential.

Proof. We just observe that all the conditions of Theorem 1 are satisfied.

Of course one cannot expect that every compact space which is the union of an infinite collection of metrizable spaces should be sequential. But we can prove the following curious assertion.

3. Theorem. If X is an infinite compact and $X = \bigcup\{X_\alpha : \alpha < \aleph_1\}$ where each X_α is metrizable then there exists in X a non-trivial converging sequence.

Proof. Clearly we may assume that every X_α is discrete. Suppose that every converging sequence in X is trivial.

By transfinite recursion we shall define a compact $F_\alpha \subset X$ for each $\alpha < \aleph_1$ in such a way that the following conditions will be satisfied: 1) if $\alpha' < \alpha'' < \aleph_1$ then $F_{\alpha''} \subset F_{\alpha'}$; 2) $|F_\alpha| \geq \aleph_0$ for every $\alpha < \aleph_1$, and 3) if $\alpha' < \alpha'' < \aleph_1$, then $X_{\alpha'} \cap F_{\alpha''} = \emptyset$.

Put $F_0 = X$. Suppose that $\alpha^* < \aleph_1$ and that for each $\alpha < \alpha^*$ a compact F_α is already defined in such a way that the conditions 1),2),3) for all these α are satisfied. We put $\Phi_{\alpha^*} = \bigcap\{F_\alpha : \alpha < \alpha^*\}$. Let us show that $|\Phi_{\alpha^*}| \geq \aleph_0$.

Let $|\Phi_{\alpha^*}| < \aleph_0$. Then, by 2), $|F_\alpha \setminus \Phi_{\alpha^*}| \geq \aleph_0$ for all $\alpha < \alpha^*$. It follows that α^* is a limit ordinal. Let us fix a sequence $\{\alpha_n : n \in N^+\}$ of ordinals converging to α^* such that $\alpha_{n'} < \alpha_{n''} < \alpha^*$ whenever $n' < n''$. It follows from 1) that $\Phi_{\alpha^*} = \bigcap\{F_{\alpha_n} : n \in N^+\}$.

As $F_{\alpha_n} \setminus \Phi_{\alpha^*}$ is infinite for each $n \in N^+$, we can choose

$x_n \in F_{\alpha_m} \setminus \Phi_{\alpha^*}$ for every $n \in N^+$ in such a way that $x_{n'} \neq x_{n''}$ when $n' \neq n''$, $n', n'' \in N^+$. If $\mathcal{U}$ is any open neighborhood of the set Φ_{α^*} in X we can find $n^* \in N^+$ such that $F_{\alpha_{n^*}} \subset \mathcal{U}$. As X is a compact, this follows from 1). Then $x_m \in \mathcal{U}$ for every $m \geq n^*$. It follows that the set $Y^* = \{x_n : x \in N^+\} \cup \Phi_{\alpha^*}$ is closed in X. Hence it is also a compact. As $|\Phi_{\alpha^*}| < \aleph_0$, $|Y^*| = \aleph_0$. Then Y^* contains a non-trivial converging sequence - contradiction. Therefore $|\Phi_{\alpha^*}| \geq \aleph_0$.

If α^* is a limit ordinal then from $\Phi_{\alpha^*} \subset F_\alpha$ and from 3) it follows that $X_\alpha \cap \Phi_{\alpha^*} = \emptyset$ for all $\alpha < \alpha^*$. Then we put $F_{\alpha^*} = \Phi_{\alpha^*}$. Let us suppose now that α^* has an immediate predecessor $\alpha^* - 1$. Then 3) implies that $X_\alpha \cap \Phi_{\alpha^*} = \emptyset$ for all $\alpha < \alpha^* - 1$.

Let us distinguish the following two cases:

a) $|X_{\alpha^*-1} \cap \Phi_{\alpha^*}| < \aleph_0$, and b) $|X_{\alpha^*-1} \cap \Phi_{\alpha^*}| \geq \aleph_0$. If a) holds then the theorem II.8 of [1] implies that $\Phi_{\alpha^*} \setminus X_{\alpha^*-1}$ is not discrete (otherwise there would exist a non-trivial converging sequence in Φ_{α^*}). Thus we can fix a non-isolated point y^* in $\Phi_{\alpha^*} \setminus X_{\alpha^*-1}$. Now we choose a neighborhood $\mathcal{U}^*$ of y^* in Φ_{α^*} such that $c\ell(\mathcal{U}^*) \cap \cap (X_{\alpha^*-1} \cap \Phi_{\alpha^*}) = \emptyset$. Then $c\ell(\mathcal{U}^*)$ is an infinite compact and, obviously, $c\ell(\mathcal{U}^*) \cap X_{\alpha^*-1} = \emptyset$. From $c\ell(\mathcal{U}^*) \subset \Phi_{\alpha^*}$ it follows also that $c\ell(\mathcal{U}^*) \cap X_\alpha = \emptyset$ for each $\alpha < \alpha^* - 1$. We put $F_{\alpha^*} = c\ell(\mathcal{U}^*)$. The condition 3) is clearly satisfied for F_{α^*}.

Let us consider now the case b). We have: $|X_{\alpha^*-1} \cap \Phi_{\alpha^*}| \geq \aleph_0$. Put $F_{\alpha^*} = c\ell(X_{\alpha^*-1} \cap \Phi_{\alpha^*}) \setminus$

$\setminus(X_{\alpha^*-1} \cap \Phi_{\alpha^*})$. As X_{α^*-1} is discrete and the set $X_{\alpha^*-1} \cap \Phi_{\alpha^*}$ is infinite, it follows that $F_{\alpha^*} \cap (X_{\alpha^*-1} \cap \Phi_{\alpha^*}) = \emptyset$. But $F_{\alpha^*} \subset \Phi_{\alpha^*}$. Hence $F_{\alpha^*} \cap X_{\alpha^*-1} = \emptyset$ and $F_{\alpha^*} \cap X_{\alpha} = \emptyset$ for each $\alpha < \alpha^* - 1$. Finally, the set F_{α^*} cannot be finite - otherwise the infinite compact $c\ell(X_{\alpha^*-1} \cap \Phi_{\alpha^*})$ would be the union of two discrete spaces - $X_{\alpha^*-1} \cap \Phi_{\alpha^*}$ and F_{α^*} By II.8 of [1], that would provide us with an infinite converging sequence. Hence $|F_{\alpha^*}| \geq \aleph_0$. The construction of the transfinite sequence $\{F_\alpha : \alpha < \aleph_1\}$ of compacts in X satisfying the conditions 1),2) and 3) is complete. Put $F^* = \bigcap\{F_\alpha : \alpha < \aleph_1\}$. From 1) and 2) it follows that $F^* \neq \emptyset$. On the other hand, 3) implies that $F^* \cap X_\alpha = \emptyset$ for each $\alpha < \aleph_1$. As $X = \bigcup\{X_\alpha : \alpha < \aleph_1\}$ and $F^* \subset X$ it follows that $F^* = \emptyset$. The contradiction we arrived at means that there exists a non-trivial converging sequence in X. Theorem 3 is proved.

4. <u>Corollary</u>. It is consistent to assert that every infinite compact X such that $|X| \leq 2^{\aleph_0}$ contains a non-trivial converging sequence.

Proof. We assume (CH). Then $X \in \mathcal{M}_{\aleph_1}$ and the conclusion follows from Theorem 3(a simple direct proof of 4 is also possible).

5. <u>Problems</u>. Let X be a compact, $X \in \mathcal{M}_\tau$ and $c(X) \leq \tau$, where $\tau > \aleph_0$. Is it true then that $d(X) \leq \tau$ or that $\pi w(X) \leq \tau$? Must X contain then a dense subspace Y such that $w(Y) \leq \tau$?

Let X be a compact and $X = X_1 \cup X_2$ where each X_i, $i = 1,2$, satisfies the following condition: for every $A \subset X_i$ such that $|A| \leq \aleph_0$ the closure of A in X_i is a space with a countable base. Is it true then that for every countable $A \subset X$ $w(c\ell(A)) \leq \aleph_0$? Is this true under the additional assumption that $\chi(X_i) \leq \aleph_0$ for each i = 1,2 ?

On the other hand we have the following example.

6. Example. Let Φ be the space of all ordinals not exceeding the first uncountable ordinal $\aleph_1$. The weight of Φ is equal to $\aleph_1$. Hence there exists a compact $\tilde{\Phi}$ such that $\tilde{\Phi} = \Phi \cup N^+$, $\Phi \cap N^+ = \emptyset$, $c\ell(N^+) = \tilde{\Phi}$ and all the points of N^+ are isolated in $\tilde{\Phi}$. We put $X = N^+ \cup (\Phi \setminus \{\aleph_1\})$ and $Y = \{\aleph_1\}$. The space X is separable as $c\ell(N^+) \supset \Phi \setminus \{\aleph_1\}$. Further $\chi(y,\Phi) \leq \aleph_0$ for every $y \in \Phi \setminus \{\aleph_1\}$. From $\Phi = \bigcap \{\tilde{\Phi} \setminus \{n\} : n \in N^+\}$ it follows that $\chi(\Phi,\tilde{\Phi}) \leq \psi(\Phi,\tilde{\Phi}) \leq \aleph_0$ (as $\tilde{\Phi}$ is a compact). By the transitivity of character in compacts (see [2]) we conclude that $\chi(y,\tilde{\Phi}) \leq \aleph_0$, for each $y \in \Phi \setminus \{\aleph_1\}$. Thus $\chi(y,X) \leq \aleph_0$ for $y \in \Phi \setminus \{\aleph_1\}$. But $X \setminus (\Phi \setminus \{\aleph_1\}) = N^+$ and all points of N^+ are isolated in X. Hence $\chi(y,X) \leq \aleph_0$ for all $y \in X$. In other words, X is separable and first countable. The singleton Y also satisfies these conditions. We have: $\tilde{\Phi} = Y \cup X$, where $t(\tilde{\Phi}) \geq t(\Phi) \geq \aleph_1$. Thus we have constructed a compact of uncountable tightness which is the union of two separable first countable spaces (one of which is open and another closed in this compact).

We shall conclude this article with the following problem, related to the problem II.13 of [1].

7. <u>Problem</u>. Let X be a compact and $X = X_1 \cup X_2$ where both X_1 and X_2 are metrizable. Is it true that the compact X is strongly Fréchet in the sense of P. Simon and D. Preiss? (see [5]).

<u>Remark</u>. In [1] we have asked whether every compact satisfying the conditions of Problem 7 must be Eberlein compact? (see [1], II.13). Of course the positive answer to this question would imply the positive answer to Problem 7 - by the result of P. Simon and D. Preiss in [5]. Hence the negative solution of Problem 7 would provide us with the negative answer to II.13.

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