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REMARKS ON SOME NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract: We prove the existence of weak solutions for boundary value problems for ordinary differential equations of second order. This improves the earlier results since there are supposed no restrictions on the null-space of the linear part of the differential operator and some new types of nonlinearities are considered.

Key words: Nonlinear ordinary differential equations, boundary value problems.

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1. Introduction. Let n be a positive integer. We consider the existence of a solution of nonlinear two point boundary value problem

(1) $$\begin{cases} -u''(x) - n^2u(x) + g(u(x)) = f(x), & x \in (0,\pi) \\ u(0) = u(\pi) = 0, \end{cases}$$

where $g(\xi)$ is a bounded continuous real valued function defined on the real line R. The typical examples of the considered nonlinearities $g(\xi)$ are the following functions:

(2) $$g(\xi) = \xi(1+\xi^2)^{-1},$$

(3) $$g(\xi) = \begin{cases} \sin(\log \xi)^{1/2}, & \xi \geq 1 \\ 0, & 0 \leq \xi < 1 \\ -g(-\xi), & \xi < 0. \end{cases}$$

For the functions of the type (2) it is known that the problem (1) has at least one weak solution $u \in W_0^{1,2}(0,\pi)$ provided

$$(4) \qquad n = 1$$

and

$$f \in L_1(0,\pi), \qquad \int_0^\pi f(x) \sin nx \, dx = 0$$

(see [3]; for partial differential analogue see [1],[5]). In the previously mentioned papers the assumption (4) is essential; here we prove the same result (see Theorem 2) also for $n = 2,3,\ldots$.

For the function $g(\xi)$ given by (3) we shall prove that the necessary and sufficient condition for the weak solvability of the problem (1) is

$$f \in L_1(0,\pi), \qquad \left|\int_0^\pi f(x) \sin nx \, dx\right| < 2$$

(see Theorem 3).

Note that both previous examples are not included in the abstract results of the Landesman-Lazer type (see the following proposition the proof of which follows from [7]; see also [2],[4],[6]).

<u>Proposition</u>. Let $g(\xi)$ be a continuous function on R with finite limit $g(+\infty) = \lim_{\xi \to +\infty} g(\xi)$. Suppose that there exists $\xi_0 \in R$ such that $g(\xi) = -g(-\xi)$ for $|\xi| \geq \xi_0$. Let $f \in L_1(0,\pi)$.

Then the boundary value problem (1) has at least one weak solution $u \in W_0^{1,2}(0,\pi)$ provided

$$\left|\int_0^\pi f(x) \sin nx \, dx\right| < 2\, g(+\infty).$$

2. Notation, terminology. In the sequel, L_1 will denote the space of all measurable real-valued functions u such that $|u|$ is integrable over $(0,\pi)$, with the usual norm

$$\|u\|_1 = \int_0^{\pi} |u(\tau)|\, d\tau \quad .$$

The symbol C denotes the space of all continuous functions u on $[0,\pi]$ with the norm

$$\|u\|_C = \sup_{\tau\in[0,\pi]} |u(\tau)| \quad .$$

Denote by $W_0^{1,2} = W_0^{1,2}(0,\pi)$ the Sobolev space of all real-valued absolutely continuous functions u on the interval $[0,\pi]$, $u(0) = u(\pi) = 0$, whose derivatives u' (which exists almost everywhere) are square-integrable. It is easy to see that $W_0^{1,2}$ is a Hilbert space with the inner product

$$\langle u,v\rangle = \int_0^{\pi} u'(\tau)v'(\tau)\,d\tau$$

and with the norm

$$\|u\|_{1,2} = \langle u,u\rangle^{1/2} \; .$$

Moreover, for arbitrary $u\in W_0^{1,2}$ it is

$$\|u\|_C \leqq \pi^{1/2}\, \|u\|_{1,2}. \tag{5}$$

Definition. Let $g(\xi)$ be continuous on R and let $f\in L_1$. The function $u\in W_0^{1,2}$ is said to be a weak solution of the boundary value problem (1) if every $v\in W_0^{1,2}$ the integral identity

$$\int_0^{\pi} u'(x)v'(x)dx - n^2 \int_0^{\pi} u(x)v(x)dx + \int_0^{\pi} g(u(x))v(x)dx =$$

$= \int_0^{\pi} f(x)v(x)dx$ holds.

3. <u>A priori estimates</u>. The following lemmas containing the a priori estimates, will be basic for the existence theorems.

<u>Lemma 1</u>. Let $\psi(\xi)$ be a bounded continuous function on R and $f \in L_1$. Then for arbitrary weak solution $u \in W_0^{1,2}$ of the problem

$$(6) \qquad \begin{cases} -u''(x) - n^2 u(x) + \psi(u(x)) = f(x), & x \in (0, \pi) \\ u(0) = u(\pi) = 0 \end{cases}$$

and for each $x \in [0, \pi]$ it is

$$|u(x) - 2\pi^{-1}\left(\int_0^{\pi} u(y) \sin ny\, dy\right) \sin nx \,| \leqq$$

$$\leqq \pi (n + 1)^2 (2n + 1)^{-1} \left(\sup_{\xi \in R} |\psi(\xi)| + \|f\|_1 \right) =$$

$$= c_1(\psi, f, n) = c_1.$$

<u>Proof</u>. Define the operators $\mathcal{L}_n$ and S on the space $W_0^{1,2}$ with the values in $W_0^{1,2}$ by the integral identities

$$\langle \mathcal{L}_n u, v \rangle = \int_0^{\pi} u'(x)v'(x)dx - n^2 \int_0^{\pi} u(x)v(x)dx,$$

$$\langle Su, v \rangle = \int_0^{\pi} \psi(u(x))v(x)dx$$

$(u.v \in W_0^{1,2})$. Denote

$$X_n = \{ v \in W_0^{1,2};\ \int_0^{\pi} v'(x) \cos nx\, dx = 0 \},$$

$$Y_n = \{ \alpha \sin nx;\ \alpha \in R \}.$$

(i) The equation

(7) $\mathcal{L}_n u = z$

is solvable for $z \in W_0^{1,2}$ iff $z \in X_n$. If $z \in X_n$ then there exists uniquely determined $u \in X_n$ which is a solution of the equation (7). Denote by K_n the mapping from X_n into X_n such that

$$K_n z = u \text{ iff } \mathcal{L}_n u = z$$

(the so-called right inverse of the operator $\mathcal{L}_n$). The mapping K_n is bounded linear operator which norm $\| K_n \|$ satisfies the inequality:

(8) $\| K_n \| \leqq (n + 1)^2 (2n + 1)^{-1}$.

(ii) It is easy to see that

(9) $\sup_{v \in W_0^{1,2}} \| Sv \|_{1,2} \leqq \pi^{1/2} \sup_{\xi \in R} | \psi(\xi) |$.

(iii) Denote by P_n the orthogonal projection of $W_0^{1,2}$ onto Y_n, i.e. $(P_n u)(x) = 2 \pi^{-1} (\int_0^{\pi} u(y) \sin ny \, dy) \sin nx$, and Q_n the orthogonal projection of $W_0^{1,2}$ onto X_n, i.e.

$$Q_n u = u - P_n u.$$

Obviously

(10) $\| Q_n \| = 1$.

(iv) If $u \in W_0^{1,2}$ is a solution of the equation

$$\mathcal{L}_n u + Su = z, \quad z \in W_0^{1,2} ,$$

then $Q_n u = -K_n Q_n (Su - z)$ and thus

(11) $\| Q_n u \|_{1,2} \leqq (n + 1)^2 (2n + 1)^{-1} (\pi^{1/2} \sup_{\xi \in R} | \psi(\xi) | +$

$+ \| z \|_{1,2})$ (see (8), (9), (10)).

(v) For $f \in L_1$ define $z \in W_0^{1,2}$ by

$$\langle z, v \rangle = \int_0^{\pi} f(x) v(x) dx, \quad v \in W_0^{1,2}.$$

Then

$$\| z \|_{1,2} \leqq \pi^{1/2} \| f \|_1.$$

Substituting the last inequality into (11) and using the imbedding inequality (5) we obtain the assertion.

Lemma 2. Let $\psi(\xi)$ be a bounded continuous function on R, $a > \xi_0 \geqq 0$, $f \in L_1$. Suppose

$$\inf_{\xi \geqq a} \psi(\xi) > 0,$$

$$\psi(\xi) = -\psi(-\xi), \quad |\xi| \geqq \xi_0,$$

and denote

$$\Gamma = \Gamma(a, \psi) = 2 \inf_{\xi \geqq a} \psi(\xi).$$

Let

$$\left| \int_0^{\pi} f(x) \sin nx \, dx \right| < \Gamma.$$

Then arbitrary weak solution $u \in W_0^{1,2}$ of the problem (6) satisfies

$$\left| 2 \pi^{-1} \int_0^{\pi} u(x) \sin nx \, dx \right| \leqq c_2(a, \psi, f, n) = c_2,$$

where

$$c_2 = (a + c_1) c_3,$$

$$c_3 = (\Gamma + \pi \sup_{\xi \in R} |\psi(\xi)|)^{1/2} \left(\Gamma - \left| \int_0^{\pi} f(x) \sin nx \, dx \right| \right)^{-1/2}.$$

Proof. Suppose that there exists a weak solution $u \in W_0^{1,2}$

of (6) such that

$$t = 2\pi^{-1}\int_0^{\pi} u(x)\sin nx\,dx > (a + c_1)\,c_3.$$

Note that $c_3 \geqq 1$. Choose $\alpha > 0$ sufficiently small and such that

$$t > (a + c_1)(c_3^{-1} - \alpha)^{-1}.$$

Put

$$\varepsilon = n^{-1}\arcsin\,(c_3^{-1} - \alpha).$$

Then

$$\left|\int_0^{\pi} f(x)\sin nx\,dx\right| < \Gamma\cos^2 n\varepsilon - \pi\sup_{\xi\in R}|\psi(\xi)|\sin^2 n\varepsilon \leqq$$

$$\leqq \Gamma\cos n\varepsilon - 2n\varepsilon\sup_{\xi\in R}|\psi(\xi)|\sin n\varepsilon =$$

$$= \inf_{\xi\geqq a}\psi(\xi)\sum_{k=0}^{n-1}\int_{\frac{\pi}{n}k+\varepsilon}^{\frac{\pi}{n}(k+1)-\varepsilon}|\sin ny|\,dy -$$

$$- 2n\varepsilon\sup_{\xi\in R}|\psi(\xi)|\sin n\varepsilon \leqq \int_0^{\pi}\psi(u(x))\sin nx\,dx =$$

$$= \int_0^{\pi} f(x)\sin nx\,dx,$$

which is a contradiction. Analogously it is possible to prove that

$$t \geqq -(a + c_1)\,c_3.$$

4. Existence results

__Theorem 1.__ Let $g(\xi)$ be a continuous bounded and odd function on R, $0 < a < b$, $b - a > 2c_1$, $f \in L_1$. Then the boundary value problem (1) has at least one weak solution provided

$$\inf_{\xi \in [a,b]} g(\xi) >$$

$$2^{-1} [\pi \sup_{\xi \in R} |g(\xi)| + (b - c_1)^2 (a + c_1)^{-2} | \int_0^\pi f(x) \sin nx \, dx |] .$$

$$\cdot [(b - c_1)^2 (a + c_1)^{-2} - 1]^{-1} .$$

Proof. Choose $a < b_1 < b$ such that the function

$$\psi(\xi) = \begin{cases} g(\xi), & |\xi| \le b_1 \\ g(b_1), & \xi > b_1 \\ -g(b_1), & \xi < -b_1 \end{cases}$$

satisfies the following conditions:

$$(12) \quad | \int_0^\pi f(x) \sin nx \, dx | < 2 \inf_{\xi \ge a} \psi(\xi) \le 2 g(b_1),$$

$$b_1 > c_1(\psi, f, n) + c_2(a, \psi, f, n).$$

Since $\psi(+\infty) = g(b_1)$ and $\psi(\xi)$ is odd, then according to the inequality (12) and with respect to Proposition (see Section 1) there exists at least one weak solution $u \in W_0^{1,2}$ of the problem (6). From Section 3 it follows that

$$|u(x)| \le b_1, \; x \in [0, \pi] .$$

Thus the function u is also a weak solution of the problem (1).

The assertions of the following two theorems we obtain immediately by verifying the assumptions of Theorem 1.

Theorem 2. Let $g(\xi)$ be a continuous bounded and odd function on R, $a > 0$. Suppose

$$\lim_{\xi \to +\infty} \xi^2 \min_{\tau \in [a, \xi]} g(\tau) = +\infty .$$

Then the boundary value problem (1) has at least one weak solution for arbitrary $f \in L_1$ with

$$\int_0^{\pi} f(x) \sin nx \, dx = 0.$$

Definition. The bounded continuous and odd function $g(\xi)$ is said to be expansive if for each p,

$$0 \leq p < \sup_{\xi \in R} g(\xi),$$

there exist sequences $0 < a_k < b_k$,

$$\lim_{k \to \infty} b_k a_k^{-1} = +\infty \quad ,$$

such that

$$\lim_{k \to +\infty} \min_{\xi \in [a_k, b_k]} g(\xi) > p.$$

Theorem 3. Let $g(\xi)$ be an expansive function. Then

$$f \in L_1, \quad \left| \int_0^{\pi} f(x) \sin nx \, dx \right| < 2 \sup_{\xi \in R} g(\xi)$$

is a necessary and sufficient condition for the boundary value problem (1) to be weakly solvable.

Remarks. Our investigation was restricted to the odd functions $g(\xi)$. By the same method (only the form of the constant c_2 is more complicated) it is possible to prove essentially the same assertions without this additional assumption. The method of present paper can be extended also for periodic problems as well as for boundary value problems for partial differential equations.

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