Commentationes Mathematicae Universitatis Carolinae

Karel Čuda

A nonstandard set theory

Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 4, 647--663

Persistent URL: http://dml.cz/dmlcz/105725

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,4 (1976)

A NONSTANDARD SET THEORY

Karel ČUDA, Praha

Abstract: The paper concerns a first order theory similar to set theory, in which one can do certain nonstandard considerations. Three metatheorems are presented which enable us to manage the relation between the standard and nonstandard in a better way. The metatheorems are applied to some concepts of calculus. From the formal point of view the theory we work in is a theory of semiset.

Key words: Standard, nonstandard, infinitely small, infinitely large, near, elementary equivalence, ideal elements, natural extension.

AMS: Primary 02K10, 02H25 Ref. Ž.: 2.641.3,2.666

Introduction. In this paper, we will consider a first order theory similar to the set theory, in which, however, one can do certain nonstandard considerations. We believe this theory is more lucid than the usual nonstandard procedures. It should be feasible to a reader who is not a specialist in logic, and enable him to understand most of the so called nonstandard considerations. Intuitively, the theory is obtained by adding a class (constant) K of all standard sets (sets which do not need "infinitely small", "infinitely large" for their existence) to the universe of sets and classes. We also require the existence of natural numbers outside of K (infinitely large natural numbers).

The situation is similar to that one of adding the imaginary unit i to real numbers. We require the class K to have three natural properties and we use these properties for nonstandard work. The principal difference between the set theory and the nonstandard set theory lies in the allowing for a subclass of a set which is not a set. Thus, we can work with "infinitely large naturals" without being forced to accept the smallest of them. The principal properties of classes are preserved. Classes are determined by their elements and we have the existence of classes defined in a "reasonable" manner. From the formal point of view the basic theory is a theory of semisets, but the reader need not be acquainted with the general theory of semisets, similarly as when working with complex numbers, one does not need vector analysis. The principal stimulations for the construction of the basic theory, not the concrete version of axioms, is due to Petr Vopěnka. Recently, an analogous (stronger) theory has appeared in the literature. That one is closer to nonstandard work in models (see H). The theorems given in the paper are well known and nonstandard proofs can be found in the literature. The theorems and proofs are given here to illustrate the usage of the basic theory and the application of the metatheorems 1,2,3. I did not find these in the literature. The nonstandard set theory was presented at the Prague seminar of Set Theory.

§ 1. Axioms

1.01. The fundamental symbols are:

$X,Y,Z,\ldots$ capitals from the end of the Roman alphabet designate variables for classes.

$x,y,z,\ldots$ lower case letters from the end of the Roman alphabet designate variables for sets.

$\in$, $=$ binary predicative symbols for membership and equality.

K a class constant.

Other logical symbols and set theoretical symbols.

Remark: In applications, we also use capital Roman letters for sets according to the common usage and we also use Greek letters for ordinal numbers.

1.02. Formulas. Formulas are constructed in the usual way from the atomic formulas $x \in Y$ etc. We use symbols φ, ψ for formulas. We write $\varphi(\vec{x},\vec{Y})$ instead of $\varphi(x_1,\ldots$ $\ldots,x_n,Y_1,\ldots,Y_m)$, meaning: Every free variable from φ occurs in x_i or Y_j and no bound variable occurs in x_i or Y_j. Thus $\vec{x} \in M$, $\vec{x} = \vec{y}$, $\vec{x} = y$ means that for every i $x_i \in M$, $x_i =$ $= y_i$, $x_i = y$ respectively.

1.03. Metadefinitions: 1) We call standard the elements in K, nonstandard those ones not in K. We call standard the concepts and the assertions which do not need K, nonstandard those ones which do need K. Thus "a is a standard set" is a nonstandard assertion.

2) The formulas without class variables are called set formulas. The formulas without bounded class variables are said to be normal.

1.0.4. Axioms: All the axioms of Z.F. theory with the axiom of choice are assumed. For classes, we accept the axioms of the G.B. theory of classes. In particular: For every normal formula $\varphi(x,\vec{X},K)$ there is a class X, such that

$$x \in X \equiv \varphi(x,\vec{X},K)$$

is provable. We designate this class by $\{x;\ \varphi(x,\vec{X},K)\}$ (as usual).

Attention: Classes defined with the help of K need not have a set intersection with every set (as it is the case in the set theory).

1.05. Definition: Class parts of sets are called semisets. A semiset not being a set is called a proper semiset. For semisets we use small Greek letters.

1.06. Further we postulate that for K the following three groups of axioms hold:

I. The axioms of the elementary equivalence (EE).

For a set formula $\varphi(\vec{x})$ we denote by $\varphi^K(\vec{x})$ the formula obtained from φ by the restriction of all the quantifiers to K. (φ^K is not a set formula, but it is normal.)

For every set formula $\varphi(\vec{x})$

$$(\forall \vec{x} \in K)(\ \varphi(\vec{x}) \equiv \varphi^K(\vec{x}))$$

is an axiom of NST (the basic theory).

II. The axiom of ideal elements (IE).

$$(\forall x \in K)(\exists\ y)(\mathrm{Fin}(y)\ \&\ x \cap K \subseteq y).$$

III. The axiom of the natural extension (NE).

$$(\forall X \subset K)((\exists x)(X \subseteq x) \longrightarrow (\exists x \in K)(X = x \cap K)).$$

Remarks: 1) EE tell us that every set formula holds in the "small world" of standard sets K and the "large (extended) world" V simultaneously. If we define a set b by a set formula with only standard parameters then b must be a standard set. Any definable relation holds simultaneously in K and in V. E.g., if we prove that for a set $x \in K$ there is a set $y \in V$ with a standard set property $\varphi(y)$, then there is a standard y with the property $\varphi(y)$. The proof of the existence of such an element in V may be easier (with the help of the nonstandardness) than the one of the existence in K.

Attention: We are not allowed to use EE for nonstandard formulas (that is, formulas having K or a set not in K as its parameter).

2) IE gives us infinitely large natural numbers. Really: Let x be a (nonstandard) finite set, such that $\omega \cap K \subset x$. As $\omega \cap K$ is the class of all natural numbers of the "small world" the number of the elements of x must be (intuitively) infinitely large. We also prove that every natural number from V - K is bigger than every natural number from K. We can expect (from the formulation of IE) that some facts holding for the finite sets may be used for infinite ones.

3) NE asserts that every semiset consisting only of standard sets determines a standard set with exactly the same standard elements. (Note that standard sets may have nonstandard elements. ω is a standard set, it is definable, by 2) there are infinitely large natural numbers - nonstandard members of ω .) We get a fruitful use of NE if we realize that relations are sets, too. We can define concepts with standard sense only for standard elements - for nonstandard ones the concepts are defined canonically by

NE. In other words: If we define some concept for the standard elements only, NE gives us a concept with standard sense (definable by a standard formula). E.g., if we define (in a nonstandard way) for every standard sequence $\{a_n\}$ of real numbers, the concept of convergence and determine which standard real number a is the limit of $\{a_n\}$, then these concepts are naturally extended for nonstandard sequences. Any standard assertion (the standard definition for example) holding for the standard sequences and their limits must hold for sequences and their limits from the natural extension. For the work with concepts defined in such a way we need not have a standard definition (nor the existence of a standard definition, for that matter).

Sometimes it is useful to work in the so called saturated models. Such a work can be done if we add some axioms or axioms schemas to our theory (see H).

Prove now that nonstandard natural numbers are bigger than standard ones.

1.07. Theorem: Let $n \in K \cap \omega$, let $m < n$. We have $m \in K$.

Proof: Put $\sigma = \{k; k < m \ \& \ k \in K\}$. Let x be the natural extension of σ. We have $(\forall k \in x)(k < n)$ (we use EE). Put $\bar{m} = \max(x) + 1$. $\bar{m}$ is standard (by EE). We have $\bar{m} - 1 \in x$ & & $\bar{m} - 1 \in K$, thus $\bar{m} - 1 < m$ and $\bar{m} \leq m$. Let $\bar{m} < m$. We have $\bar{m} \in \sigma$ (by the definition of σ), $\bar{m} \in x$ in a contradiction to $\bar{m} =$ $= \max(x) + 1$. Thus $\bar{m} = m$ and m is standard.

§ 2. The relation between standard and nonstandard definitions and assertions.

The three metatheorems given in this paragraph, enable us to manage the relation between standard and nonstandard in a better way.

2.01. <u>Metatheorem 1</u> (MT1). Let $\varphi(\vec{x},\vec{y})$ be a set formula, let $\vec{y} \in K$. The following equivalences may be proved in NST.

$$(\overrightarrow{Qx})\ \varphi(\vec{x},\vec{y}) \equiv (\overrightarrow{Q^K x})\ \varphi(\vec{x},\vec{y}) \equiv (\overrightarrow{Q^K x})\ \varphi^K(\vec{x},\vec{y})$$

Where Q_i is a quantifier ($\forall$ or $\exists$), Q_i^K is a quantifier restricted to K and φ^K is the formula got from φ by the restriction of all the quantifiers of φ to K.

Demonstration: The equivalence of the first and the third part is exactly EE. We obtain the equivalence of the second and the third part by applying successively logical laws (distributivity for quantifiers) on EE.

Remarks: 1) The MT1 states that: For a set formula with standard parameters it is immaterial if we consider it in the "standard world" K or if we restrict some quantifiers from the beginning of the formula to K or if we consider it in the "extended world" V.

2) The MT1 gives us a method for proving standard theorems. We restrict quantifiers from the beginning of the formula to K until a chosen existential quantifier. Then we prove (using the nonstandardness) the existence of the needed element in the "extended world" V. The MT1 gives the validity of the theorem. As an example we prove that the nonstandard condition for a limit implies the standard one. Before doing it, let us define the notion of nearness.

2.02. Definitions: 1) $x \mathrel{\hat{=}} 0 \equiv (\forall n \in K)(|x| < \frac{1}{m})$.

A real number x is infinitely small (near to 0) iff the absolute value of x is smaller than any positive standard real number.

2) $y \mathrel{\hat{=}} x \equiv y - x \mathrel{\hat{=}} 0$. y is near to x.

The concept of nearness has reasonable properties with regard to addition and multiplication. Similarly, we can introduce the concept of nearness for metric and normed spaces. The introduction of the nearness into uniform and topological spaces is a little more complicated.

2.03. Assertion: Let $\{a_n\}$ be a standard sequence of real numbers and let a be a standard real number. Then

$$(\forall n \in \omega - K)(a_n \mathrel{\hat{=}} a) \longrightarrow \lim_{n \to \infty} a_n = a$$

Proof: By MT1 it is sufficient to prove that the assumption implies the formula $(\forall m \in K)(\exists n_0)(\forall n > n_0)(|a_n - a| < 1/m)$. Now it is sufficient to put n_0 infinitely large and recall the definition of nearness (202) and the fact that every natural number bigger than an infinitely large natural number is infinitely large (107).

If we consider the formal record of the definition of a limit and the left hand side of the proved implication, we can notice that we have $(\forall n \in \omega - K)$ instead of $(\exists n_0)(\forall n > n_0)$. Actually $(\exists n_0)(\forall n > n_0)$ bounds only one free variable, and can be read as "for any enough large natural number n". Metatheorem 2 expresses the relation between those two kinds of quantifications.

2.04. <u>Metatheorem 2</u> (MT2): Let $\varphi(\vec{n},\vec{x})$ be a set formula ($\vec{n} \in \omega$, $\vec{x}$ need not be in K). The following two equivalences can be proved in NST:

$$(\forall \vec{n}_0 \in K)(\exists \vec{n} > \vec{n}_0)\ \varphi(\vec{n},\vec{x}) \equiv (\exists \vec{n} \notin K)\ \varphi(\vec{n},\vec{x})$$

$$(\exists \vec{n}_0 \in K)(\forall \vec{m} > \vec{n}_0)\ \varphi(\vec{n},\vec{x}) \equiv (\forall \vec{n} \notin K)\ \varphi(\vec{n},\vec{x})$$

Demonstration: By duality it is sufficient to prove only the second equivalence. Let n_0 be the smallest natural number having the property $(\forall \vec{n} > n_0)\, \varphi(\vec{n}, \vec{x})$. (If such a natural number does not exist, the equivalence holds.) Now it is sufficient to realize that both sides of the equivalence are equivalent to the standardness of n_0.

As an application of MT2 we find an equivalent for nearness

2.05. Theorem: Let x, y be real numbers, then

$$x \hat{=} y \equiv (\exists n \in \omega - K)(|x - y| < \frac{1}{m})$$

Proof: $x \hat{=} y \overset{Df}{\equiv} (\forall n \in K)(|x - y| < \frac{1}{m}) \equiv$

$$\equiv (\forall n_0 \in K)(\exists n > n_0)\ (|x - y| < \frac{1}{m}) \overset{MT2}{\equiv} (\exists n \notin K)$$

$(|x - y| < \frac{1}{m})$.

Now we have means strong enough to prove the equivalences for many standard and nonstandard assertions.

2.06. Theorem: Let $\{a_n\}$ be a standard sequence of real numbers and let a be a real number. Then

1) $\lim_{n \to \infty} a_n = a \equiv (\forall n \in \omega - K)(a_n \hat{=} a)$

2) a is a limit point of $\{a_n\} \equiv (\exists n \in \omega - K)$ $(a_n \hat{=} a)$.

Proof: 1) $(\forall n \in \omega - K)(a_n \hat{=} a) \equiv (\forall n \in \omega - K)$ $(\forall m \in K)(|a_n - a| < \frac{1}{m}) \overset{MT2}{\equiv} (\forall m \in K)(\exists n_0 \in K)(\forall n > n_0)$ $(|a_n - a| < \frac{1}{m}) \overset{MT1}{\equiv} (\forall m)(\exists n_0)(\forall n > n_0)(|a_n - a| < \frac{1}{m})$.

2) $(\exists n \in \omega - K)(a_n \hat{=} a) \equiv (\exists n \in \omega - K)(\exists m \in \omega -$ $- K)(|a_n - a| < \frac{1}{m}) \overset{MT2}{\equiv} (\forall n_0 \in K)(\forall m_0 \in K)(\exists n > n_0)$

$(\exists m>m_0)(|a_n - a| < \frac{1}{m}) \overset{MT1}{\equiv} (\forall n_0)(\forall m_0)(\exists n>n_0)$
$(\exists m>m_0)(|a_n - a| < \frac{1}{m}) \equiv (\forall m)(\forall n_0)(\exists n>n_0)$
$(|a_n - a| < \frac{1}{m})$.

2.07. __Theorem__: Let f be a standard function and M a standard set. Then

1) if $x \in M \cap K$, f is continuous in x with respect to $M \equiv (\forall y \in M)(y \triangleq x \longrightarrow f(y) \triangleq f(x))$.

2) f is continuous on $M \equiv (\forall x \in K \cap M)(\forall y \in M)(x \triangleq y \longrightarrow$
$\longrightarrow f(x) \triangleq f(y))$

3) f is uniformly continuous on $M \equiv (\forall x,y \in M)(x \triangleq y \rightarrow$
$\longrightarrow f(x) \triangleq f(y))$

Proof: 1) $(\forall y \in M)(x \triangleq y \longrightarrow f(x) \triangleq f(y)) \equiv$
$\equiv (\forall y \in M)((\exists n \in \omega - K)(|x - y| < \frac{1}{n}) \longrightarrow (\forall m \in K)$
$(|f(x) - f(y)| < \frac{1}{m})) \equiv (\forall y \in M)(\forall n \in \omega - K)(\forall m \in K).$
$(|x - y| < \frac{1}{n} \longrightarrow |f(x) - f(y)| < \frac{1}{m}) \overset{MT2}{\equiv} (\forall m \in K)$
$(\exists n_0 \in K)(\forall n>n_0)(\forall y \in M)(|x - y| < \frac{1}{n} \longrightarrow |f(x) -$
$- f(y)| < \frac{1}{m}) \overset{MT1}{\equiv} (\forall m)(\exists n_0)(\forall n>n_0)(\forall y \in M)(|x - y| <$
$< \frac{1}{n} \longrightarrow |f(x) - f(y)| < \frac{1}{m}) \equiv (\forall m)(\exists n)(\forall y \in M)(|x -$
$- y| < \frac{1}{n} \longrightarrow |f(x) - f(y)| < \frac{1}{m})$

2) $(\forall x \in K \cap M)(\forall y \in M)((\exists n \in \omega - K)(|x - y| < \frac{1}{n}) \rightarrow$
$\longrightarrow (\forall m \in K)(|f(x) - f(y)| < \frac{1}{m})) \equiv (\forall x \in K \cap M)(\forall m \in K)$
$(\forall n \in \omega - K)(\forall y \in M)(|x - y| < \frac{1}{n} \longrightarrow |f(x) - f(y)| <$
$< \frac{1}{m}) \overset{MT2}{\equiv} (\forall x \in K \cap M)(\forall m \in K)(\exists n_0 \in K)(\forall n>n_0)(\forall y \in M)$
$(|x - y| < \frac{1}{n} \rightarrow |f(x) - f(y)| < \frac{1}{m}) \overset{MT1}{\equiv} (\forall x \in M)(\forall m)$

$(\exists n)(\forall y \in M)(|x - y| < \frac{1}{n} \rightarrow |f(x) - f(y)| < \frac{1}{m})$

3) $(\forall x,y \in M)((\exists n \in \omega - K)(|x - y| < \frac{1}{n}) \rightarrow$
$\rightarrow (\forall m \in K)(|f(x) - f(y)| < \frac{1}{m})) \overset{MT2}{\equiv} (\forall m \in K)(\exists n_o \in K)$
$(\forall n > n_o)(\forall x,y \in M)(|x - y| < \frac{1}{n} \rightarrow |f(x) - f(y)| <$
$< \frac{1}{m}) \overset{MT1}{\equiv} (\forall m((\exists n)(\forall x,y \in M)(|x - y| < \frac{1}{n} \rightarrow |f(x) -$
$- f(y)| < \frac{1}{m})$.

2.08. <u>Theorem</u>: Let $\{f_n\}$ be a standard sequence of functions. Let M be a standard set and f a standard function. We have

1) $\{f_n\}$ converges to f on M pointwise $\equiv$
$\equiv (\forall x \in M \cap K)(\forall n \in \omega - K)(f_n(x) \hat{=} f(x))$.

2) $\{f_n\}$ converges to f on M uniformly $\equiv$
$\equiv (\forall x \in M)(\forall n \in \omega - K)(f_n(x) \hat{=} f(x))$.

Proof: We use MT1 and MT2 in the same way as in the proofs of (206) and (207).

<u>Attention</u>: The assumption of standardness of f,x,M,a, $\{a_n\}$, $\{f_n\}$ is substantial, since $x \hat{=} y$ is not a standard formula and we use EE in the proof.

It is useful to generalize the notion of infinitely small.

2.09. Definition: Let R(x,y) be a standard binary set relation. A y is said to be R-infinitely small (RIS(y)) iff for every standard element x of $\mathcal{D}$(R) we have R(x,y). (Infinitely large natural numbers are < infinitely small.)

Now, a question arises which relations have infinitely small elements.

2.10. <u>Theorem</u>: Let R be a standard binary set relation. The following statements are equivalent:

1) There is an RIS element.

2) For every finite standard subset $\{x_i\}$ of the domain of R there is a y such that $(\forall i)R(x_i,y)$.

Proof: 1) ⟶ 2) trivial.

2) ⟶ 1) By EE 2) holds also for nonstandard finite sets of the domain of R. By IE we have a finite set D such that $\mathcal{D}(R) \cap K \subseteq D \subseteq \mathcal{D}(R)$. Let y be the element corresponding to D. y is RIS.

Remark: Relations having the property 2) are called concurrent in the literature.

Another natural question arises: Can we find an analogue of MT2 concerning the extended notion of infinitely small? The saturated models show that we can add the exact reformulation of MT2 for any standard transitive relation as a new axiom schema. In our theory we can prove the similar metatheorem.

2.11. <u>Metatheorem 3</u> (MT3): Let R be a standard transitive relation with infinitely small elements. Let $\varphi(\vec{x},\vec{y})$ be a set formula and $\vec{y}$ be standard. The following equivalences are provable in NST:

$$(\forall \vec{x}_0 \in K)(\exists \vec{x}, R(\vec{x}_0,\vec{x}))\ \varphi(\vec{x},\vec{y}) \equiv (\exists \vec{x},\ RIS(\vec{x}))\ \varphi(\vec{x},\vec{y})$$

$$(\exists \vec{x}_0 \in K)(\forall \vec{x}, R(\vec{x}_0,\vec{x}))\ \varphi(\vec{x},\vec{y}) \equiv (\forall \vec{x}, RIS\ (\vec{x}))\ \varphi(\vec{x},\vec{y})$$

Demonstration: In view of the duality it is sufficient to prove only the first equivalence. By MT1 the left hand side of the equivalence is equivalent to the following:

$(\forall \vec{x}_0)(\exists \vec{x}, R(\vec{x}_0,\vec{x}))\ \varphi\ (\vec{x},\vec{y})$

Let x_0 be RIS. Put $\vec{x}_0 = x_0$. Let $\vec{x}$ be such that $R(\vec{x}_0,\vec{x})$ and $\varphi\ (\vec{x},\vec{y})$; then $\vec{x}$ is RIS (transitivity of R). On the other hand, let there be an $\vec{x}$ such that RIS($\vec{x}$) and $\varphi\ (\vec{x},\vec{y})$. The left hand side of the equivalence holds trivially.

Let us now generalize the notion of nearness. In the standard way, nearness is described with the help of neighbourhood filters. Thus it is natural to formulate

2.12. Definition: Let T be a standard topological space and x a standard point in T. Denote by $\{U(x)\}$ the neighbourhood filter of x. Put

$$\mu(x) = \bigcap (\{\mathcal{U}(x)\} \cap K)$$

(The semiset of all points near to x.)

Note that "to be near to" need not be symmetric but must be reflexive and transitive.

We are able to prove an analogue of the theorem 205.

2.13. <u>Theorem</u>: Let R be the relation of inclusion $u \supset v$ on the neighbourhood filter of a standard point x of a standard topological space T. We have:

$y \in \mu(x) \equiv (\exists \mathcal{U}, \mathrm{RIS}(\mathcal{U}))(y \in \mathcal{U})$

Proof: $\leftarrow$ trivial

$\rightarrow$ Let $y \in \mu(x)$, let D be a finite subset of the neighbourhood filter $\{U(x)\}$ such that $\{U(x)\} \cap K \subseteq D \subseteq \{U(x)\}$. Put $U = \bigcap \{V \in D;\ V \ni y\}$. U is the required RIS neighbourhood of x.

In a similar manner we can treat uniform spaces. Using MT1 and MT3 we can easily get the topological and uniform analogues of the assertions (206) - (208).

For some nonstandard formulas, the standard equivalents are not given by the mentioned metatheorems. In the mentioned metatheorems the possibility of representation of complicated standard objects by simpler nonstandard ones is not obtained. Using nonstandardness we can represent e.g. the "Dirac function" as a continuous nonstandard function (having an infinitely small support), neighbourhood filter of a standard point x by the monad of x. Similarly, for a standard filter $\mathcal{F}$ we formulate the following

2.14. Definition: Let $\mathcal{F}$ be a standard filter on a standard set M. Put

$$\mathrm{Ker}\ (\mathcal{F}) = \bigcap\ (\mathcal{F} \cap K)$$

(the kernel of $\mathcal{F}$)

We can easily prove

2.15. Assertion: Let $\mathcal{F}$ be a standard filter on a standard set M. Then $\mathcal{F}$ is the natural extension of $\{Y \in K;\ Y \subseteq M\ \&\ Y \supseteq \mathrm{Ker}\ (\mathcal{F})\}$.

Thus, filters (parts of $\mathcal{P}(M)$) can be represented by kernels (parts of M). The situation is more interesting in the case of ultrafilters. Let M be a standard set, x an element (possibly nonstandard) of M. Put $\xi = \{A \in \mathcal{P}(M);\ x \in A\ \&\ A \in K\}$. ξ has the following four properties:

1) $A \in \xi\ \&\ B \supseteq A\ \&\ B \in K \cap \mathcal{P}(M) \longrightarrow B \in \xi$

2) $A \in \xi\ \&\ B \in \xi \longrightarrow A \cap B \in \xi$

3) $A \in K \cap \mathcal{P}(M) \longrightarrow A \in \xi \vee M - A \in \xi$

4) $0 \notin \xi$

Thus, we see that the natural extension of ξ is a standard ultrafilter on M and x is a member of his kernel. We have correspondence between standard ultrafilters on a standard set M and (possible nonstandard) members of M. This correspondence leads to a nonstandard equivalent of the compactness of topological spaces.

2.16. <u>Theorem</u>: Let T be a standard topological space. T is compact iff for every point y of T there is a standard point x of T such that $y \in \mu(x)$.

Proof: Recall the mentioned correspondence between points of T and standard ultrafilters and the correspondence between monads and neighbourhood filters of standard points of T (2.12, 2.14, 2.15). The property in the theorem is equivalent to: "For any standard ultrafilter $\mathcal{F}$ there is a standard neighbourhood filter $\{U(x)\}$ of a standard point x which is a part of $\mathcal{F}$ ". By EE we can cancel the words standard and we get an equivalent of compactness.

The theorem shows that the compactness is natural from the nonstandard point of view and leads e.g. to a very easy proof of the Tichonov theorem. (See R .)

<u>Notes</u>:

1) A nonstandard theory of measure and Lebesgue integration was also developed. It differs from the one in (BW). The main differences are in the following points:

a) The standard measure theoretic theorems are not used.

b) Instead of extending the Lebesgue measure, we define the Lebesgue measure directly in the nonstandard way.

c) Some principal theorems of measure theory and integration are proved.

2) By considerations of the presented theory and of the

nonstandard semiset theory, the following interesting statement was obtained. If ZFC + (existence of inaccessibles) is consistent, we can consistently use the following assertion at nonstandard work. Let I,J be bounded intervals of real numbers. Let $\{I_i\}$, $\{J_j\}$ be fine partitions of I,J respectively. Let the differential intervals I_i be of the same order (that is $\neg(\exists i_1,i_2)(\mu(I_{i_1})/\mu(I_{i_2}) \triangleq 0))$. Let F be a semiset (external) one-to-one correspondence between partitions $\{I_i\}$, $\{J_j\}$ such that:

a) $(\forall i)(\mu(I_i)/\mu(F(I_i)) \triangleq 1)$

b) For the definition of F we use only a (internal) set and the notion of $\triangleq$ for real numbers.

Then we have $\mu(I)/\mu(J) \triangleq 1$.

It is possible that the mentioned assertion is provable. My attempts to prove it or to prove the consistence of its negation was unsuccessful.

These results will appear elsewhere.

References

[BW] Allen R. BERNSTEIN and Frank WATTENBERG: Nonstandard measure theory, Applications of Model Theory to Algebra, Analysis, and Probability, New York 1969.

[H] Karel HRBÁČEK: Axiomatic foundations for nonstandard analysis, The City College of Cuny, New York (preprint).

[R] Abraham ROBINSON: Nonstandard analysis, North-Holland Publishing Company, Amsterdam 1966.

[VH] Petr VOPĚNKA; Petr HÁJEK: The theory of semiset, Academia, Praha 1972.

Matematický ústav
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 31.5.1976)