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AN ITERATION METHOD FOR NONLINEAR SECOND ORDER EVOLUTION EQUATIONS

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Abstract: In this paper we consider initial value problems of the type

$$u'' + Au' + Bu = 0,\ u(0) = a,\ u'(0) = b,$$

where A and B are in general nonlinear operators in Hilbert spaces satisfying certain monotonicity and continuity conditions. We show that it is posssible to solve such problems using an iteration method which requires at each step the solution of a linear initial value problem. Under somewhat more restrictive assumptions on A and B we deal also with the determination of periodic solutions of second order evolution equations by iteration.

Key words: Second order evolution equations, iteration, initial value problems, periodic solutions.

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Let X be a Hilbert space and let X^* be its dual. Furthermore, let Λ be a maximal monotone subset of $X \times X^*$ and $A \in (X \longrightarrow X^*)$ a strongly monotone Lipschitz continuous operator. In [4], Gajewski and Gröger introduced an iteration method for the solution of problems of the form

$$\Lambda u + Au \ni 0,\ u \in D(\Lambda),$$

The results contained in this paper were first presented by the author at the Summer School on "Nonlinear Analysis and Mechanics", September 1974, Stará Lesná near Poprad, Slovakia.

and they applied this method to some first oder evolution equations in Hilbert space. The purpose of this paper is to show how the iteration method can be applied to non-linear second order evolution equations in Hilbert space. We shall treat both initial value problems and the problem of determination of periodic solutions.

1. <u>The iteration method</u>. In this section we shall recall briefly the main result of Gajewski-Gröger [4]. Let X be a real Hilbert space and let X^* be its dual. The value of $f \in X^*$ at $u \in X$ will be denoted by $\langle f,u \rangle$. We define the norm on the Cartesian product $X \times X^*$ by

$$\| [u,f] \|_{X \times X^*} = (\|u\|_X^2 + \|f\|_{X^*}^2)^{\frac{1}{2}} \; \forall [u,f] \in X \times X^*.$$

Let L be the duality map of X, i.e., the mapping from X to X^* determined by

$$\langle Lu,u \rangle = \|u\|_X^2 = \|Lu\|_{X^*}^2 \quad \forall\, u \in X.$$

Since X is a Hilbert space, L is linear.

We assume that we are given operators Λ and A such that

(1) $\Lambda \subset X \times X^*$ is a maximal monotone set:

$A \in (X \longrightarrow X^*)$ is strongly monotone and Lipschitzian,

(2) i.e., $\langle Au - Av, u - v \rangle \geqq m \|u - v\|_{X^*}^2$,

$m > 0$, $\|Au - Av\|_{X^*} \leqq M \|u - v\|_X \quad \forall\, u, \forall\, v \in X.$

As usual (see e.g. Brezis [1]), we regard Λ as a multi-valued mapping from X to X^*. We define

$\Lambda u = \{f \mid f \in X^*, [u,f] \in \Lambda\}$ $\forall u \in X$, $D(\Lambda) =$

$= \{u \mid u \in X, \Lambda u \neq \emptyset\}$.

It is well known (cf. Browder [2]) that the problem

$$\Lambda u + Au \ni 0, \quad u \in D(\Lambda), \tag{3}$$

has a unique solution u provided that (1) and (2) are satisfied. In [4], Gajewski and Gröger proved the following result which shows that the solution of (3) can be obtained by an iteration method.

<u>Theorem 1</u>. Let the assumptions (1) and (2) be satisfied. Furthermore, let $0 < r < \frac{2m}{M^2}$. Then the operator $Q_r \in (X \times X^* \longrightarrow X \times X^*)$ defined by

$Q_r([u,f]) = [v, rg] \Longleftrightarrow rg + Lv = Lu - rAu, \ g \in \Lambda v,$

is contractive with the contraction constant

$k_r = (1 - 2mr + M^2r^2)^{\frac{1}{2}}$. If $f + Au = 0$, $f \in \Lambda u$, and if $([u_i, f_i])$ is determined by

$$\begin{cases} rf_i + Lu_i = Lu_{i-1} - rAu_{i-1}, \ f_i \in \Lambda u_i, \\ i = 1,2,\dots, \ u_0 \in X \text{ arbitrary}, \end{cases} \tag{4}$$

then $[u_i, f_i] \longrightarrow [u,f]$ in $X \times X^*$; more precisely

$$(\|u_i - u\|_X^2 + \|r(f_i - f)\|_{X^*}^2)^{\frac{1}{2}} \leq \frac{k_r^i}{1 - k_r} \|u_1 - u_0\|_X. \tag{5}$$

<u>Remark 1</u>. The operator Q_r in Theorem 1 is well defined. This follows easily from the fact that $r\Lambda + L$ is a one-to-one mapping from $D(\Lambda)$ onto X^*.

<u>Remark 2</u>. If assumption (2) is satisfied and

$0 < r < \frac{2m}{M^2}$, then $U_r = I - rL^{-1}A$ (I identity map of X) is contractive in X with the contraction constant $k_r = (1 - 2mr + M^2r^2)^{\frac{1}{2}}$. The fixed point u of U_r is the unique solution of $Au = 0$ (see e.g. Browder-Petryshyn [3]). Theorem 1 may be considered as a generalization of this result to problems of type (3).

Remark 3. If A is a potential operator, then the upper bound for r and the contraction constant k_r in Theorem 1 may be replaced by $\frac{2}{M}$ and $q_r = \max(1 - mr, Mr - 1)$, respectively (cf. Gajewski-Gröger [4], Remark 2).

2. Second order evolution equations

2.1. Preliminaries. Let U, V and H be Hilbert spaces such that U and V are continuously and densely imbedded into H. Moreover, let $U \cap V$ be a dense subset of U and of V. For the sake of simplicity we assume that $U \cap V$ is separable. We shall identify H with its dual space H^*. Then the following inclusion relations hold:

$$U \cap V \subset U \subset H \subset U^* \subset (U \cap V)^* ,$$

$$U \cap V \subset V \subset H \subset V^* \subset (U \cap V)^* ;$$

here U^*, V^* and $(U \cap V)^*$ denote the dual spaces of U, V and $U \cap V$, respectively. If E is one of the spaces $U \cap V$, U, V, H, then the value of $z \in E^*$ at $x \in E$ will be denoted by (z,x). The norms on U, V, H and V^* we shall denote by $|||\cdot|||$, $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ respectively.

Let $S = [0,T] \subset R^1$ be a finite interval. If E is a

Banach space, we denote by $L^2(S;E)$ the space of square integrable functions defined on S with values in E, provided with the usual norm

$$\|u\|_{L^2(S;H)} = \left(\int_S \|u(t)\|_E^2 \, dt \right)^{\frac{1}{2}} .$$

By $C^k(S;E)$ we denote the space of E-valued functions on S which have continuous derivatives up to the order k. Instead of $C^0(S;E)$ we write $C(S;E)$.

If $u \in L^2(S;H)$, we denote by u' and u'' the first and the second derivative of u in the sense of distributions over $]0,T[$ with values in $(U \cap V)^*$.

Let $X = L^2(S;V)$, $Y = L^2(S;U)$ and accordingly $X^* = L^2(S;V^*)$, $X^* = L^2(S;U^*)$. As in the preceding section we denote by L the duality map of X. The duality map of Y will be denoted by K. Let

(6) $$W = \{u \mid u \in Y,\ u' \in X,\ u'' + Ku \in X^*\}$$

and

(7) $$\|u\|_W = \left(\|u\|_Y^2 + \|u'\|_X^2 + \|u'' + Ku\|_{X^*}^2 \right)^{\frac{1}{2}} \quad \forall\, u \in W.$$

It is easy to see that W, provided with the norm $\|\cdot\|_W$ (and the corresponding scalar product), is a Hilbert space.

<u>Lemma 1</u>. Let

(8) $$D = \{u \mid u \in C^\infty(S;U),\ u' \in C^\infty(S;V),\ u'' + Ku \in C^\infty(S;V^*)\}.$$

Then D is dense in W.

Proof. Let $1 > \varepsilon > 0$, $\eta = \dfrac{\varepsilon T}{2(1-\varepsilon)}$,

$S_\varepsilon = [-\eta, T+\eta]$, $u \in W$ and

$$u_\varepsilon(t) = u((1-\varepsilon)t + \varepsilon \tfrac{T}{2}) \quad \forall t \in S_\varepsilon .$$

Then $u_\varepsilon \in L^2(S_\varepsilon ;U)$, $u'_\varepsilon \in L^2(S_\varepsilon ;V)$, $u''_\varepsilon + K_\varepsilon u_\varepsilon \in L^2(S_\varepsilon ;V^*)$, where K denotes the duality map of $L^2(S_\varepsilon ;U)$. It is easy to see that the functions u_ε (restricted to S) are converging to u in W if $\varepsilon \longrightarrow 0$. Therefore, to prove the lemma it is sufficient to show that functions of type u_ε may be approximated in W by functions from D.

Let φ be an infinitely differentiable function on R^1 with compact support such that $\varphi(-t) = \varphi(t)$ and $\int_{R^1} \varphi(t)dt = 1$. Let

$$\varphi_n(t) = n\varphi(nt), \; n = 1,2,\ldots .$$

If n is large enough we can define $u_{\varepsilon n}$ by $u_{\varepsilon n} = \varphi_n * u_\varepsilon$. By standard arguments (see e.g. [7], Lemma 1.12, Chap. IV) we obtain $u_{\varepsilon n} \in D$ and $u_{\varepsilon n} \longrightarrow u$ in W. This proves the lemma.

<u>Proposition 1</u>. The space W defined by (6) and (7) is continuously imbedded into $Z = C^1(S;H) \cap C(S;U)$, provided with the norm

$$\|u\|_Z = \sup_{t\in S} \left(|u'(t)|^2 + |||u(t)|||^2 \right)^{\frac{1}{2}} \quad \forall u \in Z.$$

<u>Proof</u>. Let J be the duality map of U. Then

$$(Ku)(t) = Ju(t) \quad \forall u \in Y, \; \forall t \in S.$$

Obviously, if $u, v \in D$ (D defined by (8)) and $t, t_0 \in S$, then

$$\int_{t_0}^{t} \{((u'' + Ku)(s), v'(s)) + ((v'' + Kv)(s), u'(s))\}\, ds$$

(9) $= (u'(t), v'(t)) - (u'(t_0), v'(t_0))$
$\quad + (Ju(t), v(t)) - (Ju(t_0), v(t_0)).$

Let $\varphi \in C^2(S)$, $\varphi(0) = \varphi'(0) = \varphi'(T) = 0$, $\varphi(T) = 1$, and let $u \in D$, $v = \varphi u$. We apply (9) to u, v and u, u - v, respectively. This yields

$$\int_0^t \{((u'' + Ku)(s),v'(s)) + ((v'' + Kv)(s),u'(s))\} ds$$

$$= (u'(t),v'(t)) + (Ju(t),v(t))$$

and

$$\int_t^T \{((u'' + Ku)(s),u'(s) - v'(s)) + ((u'' - v'' + Ku - Kv)(s), u'(s))\} ds$$

$$= - (u'(t),u'(t) - v'(t)) - (Ju(t),u(t) - v(t)).$$

By subtraction and some elementary calculations we obtain

$$|u'(t)|^2 + |||u(t)|||^2$$

$$= \int_S \{((u'' + Ku)(s),v'(s)) + ((v'' + Kv)(s),u'(s))\} ds$$

$$- 2\int^T ((u'' + Ku)(s),u'(s))ds$$

$$= \int_S \{(2\varphi(s)(u'' + Ku)(s) + \varphi'(s)u'(s), u'(s)) +$$

$$+ (Ju(s),\varphi'(s)u(s))\} ds - 2\int_t^T ((u'' + Ku)(s),u'(s))ds.$$

Consequently,

$$(10) \qquad |u'(t)|^2 + |||u(t)|||^2 \leq C^2 \|u\|_W^2 ,$$

where C is a constant not depending on t and on u.

Let now $u \in W$ be arbitrary. Because of Lemma 1 we can choose a sequence (u_n), $u_n \in D$, converging in W to u. Using (10) we obtain

$$(11) \qquad \|u_n - u_m\|_Z \leq C\|u_n - u_m\|_W .$$

From (11) follows that $u \in Z$ and

$$\|u\|_Z \leq C\|u\|_W .$$

This completes the proof.

Corollary 1. The formula (9) is valid for arbitrary $u, v \in W$. As a special case we obtain

$$\int_0^t ((u'' + Ku)(s), u'(s))ds$$

$$(12) \quad = \frac{1}{2}\left(|u'(t)|^2 + |||u(t)|||^2 - |u'(0))|^2 - \right.$$

$$\left. - |||u(0)|||^2 \right) \qquad \forall\ u \in W.$$

Remark 4. The "energy equation" (12) is closely related to a similar result obtained by Lions-Strauss [10] (Lemma 2.1). Our proof may be regarded as a modification of the proof given by Lions-Strauss.

Lemma 2. Let U be continuously imbedded into V. If $u \in W$ such that $u(0) = u(T)$, $u'(0) = u'(T)$, then

$$(13) \quad \|u\|_Y^2 \leq c(\|u'\|_X^2 + \|u'' + Ku\|_{X^*}^2), \quad c = \text{const} .$$

Proof. By the assumption of Lemma 2 we have $Y \subset X \subset L^2(S;H) \subset X^* \subset Y^*$. Let $u_n \in D$ such that $u_n \to u$ in W. Then

$$\|u_n\|_Y^2 = \int_S ((u_n'' + Ku_n - u_n'')(s), u_n(s))ds$$

$$= \int_S \{((u_n'' + Ku_n)(s)) + |u_n'(s)|^2\} ds$$

$$- (u_n'(T), u_n(T)) + (u_n'(0), u_n(0))$$

$$\leq \|u_n'' + Ku_n\|_{Y^*} \|u_n\|_Y + \|u_n'\|^2_{L^2(S;H)}$$

$$- (u_n'(T), u_n(T)) + (u_n'(0), u_n(0))$$

$$\leq c_1 (\| u_n'' + Ku_n \|_{X^*} \| u_n \|_Y + \| u_n' \|_X^2)$$

$$- (u_n'(T), u_n(T)) + (u_n'(0), u_n(0)), \quad c_1 = \text{const} .$$

Passing to the limit we find

$$\| u \|_Y^2 \leq c_1 (\| u'' + Ku \|_{X^*} \| u \|_Y + \| u' \|_X^2)$$

$$\leq \frac{1}{2} c_1^2 \| u'' + Ku \|_{X^*}^2 + \frac{1}{2} \| u \|_Y^2 + c_1 \| u' \|_X^2 .$$

This proves the lemma.

In what follows we assume as in Section 1 that we are given an operator $A \in (X \longrightarrow X^*)$ which satisfies the condition (2).

2.2. Initial value problems. We consider the problem

(14) $\quad u'' + Au' + Ku = 0, \; u \in W, \; u(0) = a, \; u'(0) = b,$

where $a \in U$ and $b \in H$ are given initial values. Let

(15) $\quad \Lambda = \{ [u', u'' + Ku] \mid u \in W, \; u(0) = a, \; u'(0) = b \} .$

Then (14) can be written as

$$\Lambda v + Av \ni 0, \; v \in D(\Lambda).$$

Lemma 3. The set $\Lambda \subset X \times X^*$ defined by (15) is maximal monotone.

Proof. Let $[u_j', u_j'' + Ku_j] \in \Lambda$, $j = 1,2$, and let $u = u_1 - u_2$. Using (12) we obtain

$$\langle u'' + Ku, u' \rangle = \frac{1}{2} (| u'(T) |^2 + ||| u(T) |||^2) \geq 0,$$

i.e., Λ is monotone. In order to prove the maximality of

Λ it is sufficient to show that $\Lambda + L$ is surjective (see e.g. Browder [2]). By definition of Λ the mapping $\Lambda + L$ is surjective if and only if the initial value problem

$$(16) \qquad u'' + Ku + Lu' = f, \; u \in W, \; u(0) = a, \; u'(0) = b,$$

is solvable for every $f \in X^*$. The solvability of (16) follows from a general existence theorem of Lions-Strauss [10]. Thus the lemma is proved.

<u>Theorem 2</u>. Let assumption (2) be satisfied. Then the problem (14) has a unique solution u for every $a \in U$ and $b \in H$. If $0 < r < \frac{2m}{M^2}$ and if (u_i) is determined by

$$(17) \quad \begin{cases} r(u_i'' + Ku_i) + Lu_i' = Lu_{i-1}' - rAu_{i-1}', \; u_i \in W, \\ u_i(0) = a, \; u_i'(0) = b, \; i = 1,2,\dots, \; u_0' \in X \text{ arbitrary}, \end{cases}$$

then

$$(18) \quad (\| u_i' - u' \|_X^2 + \| r(u_i'' + Ku_i - u'' - Ku) \|_{X^*}^2) \leqq$$

$$\leqq \frac{k_r^i}{1 - k_r} \| u_1' - u_0' \|_X$$

and

$$(19) \qquad \| u_i - u \|_Z \leqq \frac{1}{\sqrt{r}} \frac{k_r^i}{1 - k_r} \| u_1' - u_0' \|_X ,$$

where $k_r = (1 - 2mr + M^2r^2)^{\frac{1}{2}}$.

<u>Proof</u>. With the exception of (19) the assertions of the theorem follow immediately from Theorem 1 and Lemma 3. Using (12) with $u_i - u$ instead of u we find

$$| u_i'(t) - u'(t) |^2 + ||| u_i(t) - u(t) |||^2$$

$$= 2 \int_0^t ((u_i'' - u'' + Ku_i - Ku)(s), u_i'(s) - u'(s))ds$$

$$\leqq \frac{1}{r} (\| u_i' - u' \|_X^2 + \| r(u_i'' - u'' + Ku_i - Ku) \|_{X^*}^2).$$

Together with (18) this implies the assertion (19).

Remark 5. The iteration method (17) requires at each step the solution of a linear initial value problem.

2.3. Periodic solutions. In this section we assume that U is continuously imbedded into V. We consider the problem

(20) $u'' + Au' + Ku = 0, \ u \in W, \ u(0) = u(T), \ u'(0) = u'(T).$

Let

(21) $\Lambda = \{ [u', u'' + Ku] \mid u \in W, \ u(0) = u(T), \ u'(0) = u'(T) \}.$

Then (20) can be written as

$$\Lambda v + Av \ni 0, \ v \in D(\Lambda).$$

Lemma 4. The set $\Lambda \subset X \times X^*$ defined by (21) is maximal monotone.

Proof. The monotonicity of Λ follows immediately from (12). To prove the maximality of Λ we shall show that $\Lambda + L$ is surjective, i.e., that the problem

(22) $u'' + Ku + Lu' = f, \ u \in W, \ u(0) = u(T), \ u'(0) = u'(T),$

is solvable for every $f \in X^*$.

Let (H_n) be a sequence of finite dimensional subspaces of U such that $\bigcup_n H_n$ is dense in U. Identifying

$L^2(S;H_n)$ with its dual we define $K_n \in (L^2(S;H_n) \longrightarrow \longrightarrow L^2(S;H_n))$, $L_n \in (L^2(S;H_n) \longrightarrow L^2(S;H_n))$ and $f_n \in \in L^2(S;H_n)$ by

$$\langle K_n u, v \rangle = \langle Ku, v \rangle \qquad \forall\, v \in L^2(S;H_n),$$

$$\langle L_n u, v \rangle = \langle Lu, v \rangle \qquad \forall\, v \in L^2(S;H_n),$$

$$\langle f_n, v \rangle = \langle f, v \rangle \qquad \forall\, v \in L^2(S;H_n).$$

The problem

$$(23) \quad \begin{cases} u_n'' + K_n u_n + L_n u_n' = f_n, \; u_n \in L^2(S;H_n), \\ \quad u_n' \in L^2(S;H_n), \\ u_n(0) = u_n(T), \; u_n'(0) = u_n'(T), \end{cases}$$

has a unique solution u_n (see e.g.[7], Theorem 1.3, Chap. VII). Multiplying (23) by u_n' yields

$$\langle u_n'' + K_n u_n + L_n u_n', u_n' \rangle = \langle u_n'' + Ku_n + Lu_n', u_n' \rangle$$

$$= \langle Lu_n', u_n' \rangle = \| u_n' \|_X^2 = \langle f_n, u_n' \rangle = \langle f, u_n' \rangle \; .$$

Hence

$$\| u_n' \|_X \leqq \| f \|_{X^*} \; .$$

From the assumption that U is continuously imbedded into V it follows that $X^* \subset Y^*$. Therefore, multiplying (23) by u_n we obtain

$$\langle u_n'' + Ku_n + Lu_n', u_n' \rangle = - \| u_n' \|^2_{L^2(S;H)} + \| u_n \|_Y^2$$

$$= \langle f, u_n \rangle \leqq \| f \|_{Y^*} \| u_n \|_X \; .$$

Consequently,

$$\sup_n \| u_n \|_Y < \infty \; .$$

Let (n_i) be a sequence such that

$$u_{n_i} \longrightarrow u \text{ in } Y,$$

$$u'_{n_i} \longrightarrow u' \text{ in } X.$$

By standard arguments one can prove that u is a solution of (22). This completes the proof of Lemma 4.

Remark 6. The solvability of (22) can be proved also by "elliptic regularization" (see Lions [8], Chap. 4, § 7, where a special case is treated).

We are now able to apply Theorem 1 to the problem (20).

Theorem 3. Let the assumption (2) be satisfied. Then the problem (20) has a unique solution. If $0 < r < \frac{2m}{M^2}$ and if the sequence (u_i) is determined by

$$(24) \quad \begin{cases} r(u_i'' + Ku_i) + Lu_i' = Lu_{i-1}' - rAu_i', \ u_i \in W, \\ u_i(0) = u_i(T), \ u_i'(0) = u_i'(T), \ i = 1,2,\ldots, \\ u_o' \in X \text{ arbitrary,} \end{cases}$$

then

$$(25) \quad (\| u_i' - u' \|_X^2 + \| r(u_i'' + Ku_i - u'' - Ku) \|_{X^*}^2)^{\frac{1}{2}} \leq$$

$$\leq \frac{k_r^i}{1 - k_r} \| u_1' - u_o' \|_X$$

and

$$(26) \qquad \| u_i - u \|_Z \leq ck_r^i, \ c = \text{const} ,$$

where $k_r = (1 - 2mr + M^2r^2)^{\frac{1}{2}}$.

Proof. The assertion (25) follows from Theorem 1 and

Lemma 4. The assertion (26) follows from (25), (13) and Proposition 1.

<u>Remark 7</u>. The iteration method (24) requires at each step the solution of a linear problem.

2.4. <u>Further initial value problems</u>. We consider the problem

$$(27)\qquad u'' + Au' + Bu = 0,\ u' \in X,\ u(0) = a,\ u'(0) = b,$$

where $a \in V$ and $b \in H$ are given initial values and B is a Lipschitz continuous mapping from X to X^*; more precisely, let

$$(28)\qquad \| Bu - Bv \|_{X^*} \leq M_B \| u - v \|_X \qquad \forall u,\ \forall v \in X.$$

<u>Definition</u> (cf.[7], Def. 1.1, Chap. V). A mapping $C \in (X \longrightarrow X^*)$ is said to be a Volterra operator if for every $t \in S$

$u(s) = v(s)$ a.e. on $[0,t] \Longrightarrow (Cu)(s) = (Cv)(s)$ a.e. on $[0,t]$.

In what follows we shall assume that A and B are Volterra operators. We define $R \in (X \longrightarrow X)$ by

$$(29)\qquad (Rv)(t) = a + \int_0^t v(s)ds \qquad \forall v \in X.$$

Obviously, the problem (27) can be written as

$$(30)\qquad v' + (A + BR)v = 0,\ v \in X,\ v(0) = b.$$

In general $A + BR \in (X \longrightarrow X^*)$ is not strongly monotone. Therefore, it is impossible to apply Theorem 1 to (30) directly. It turns out to be useful to introduce the following norms on X and X^*, respectively:

$$\| u \|_{X_\lambda} = \left(\int_S \| e^{-\lambda s} u(s) \|^2 ds \right)^{\frac{1}{2}} \qquad \forall u \in X,$$

$$\| f \|_{X^*_\lambda} = \left(\int_S \| e^{-\lambda s} f(s) \|_*^2 ds \right)^{\frac{1}{2}} \qquad \forall f \in X^*;$$

here λ denotes a positive parameter. Using these norms we shall write X_λ and X^*_λ instead of X and X^*. The pairing between X^*_λ and X_λ is given by

$$\langle f, u \rangle_\lambda = \int_S e^{-2\lambda s} (f(s), u(s)) ds.$$

The duality maps of X_λ and X are equal.

Lemma 5. Let A and B be Volterra operators which satisfy the conditions (2) and (28), respectively. Furthermore, let R be defined by (29), Then for $u, v \in X$ we have

$$\langle Au - Av, u - v \rangle_\lambda \geqq m \| u - v \|^2_{X_\lambda} ,$$

$$\| Au - Av \|_{X^*_\lambda} \leqq M \| u - v \|_{X_\lambda} ,$$

$$\| Bu - Bv \|_{X^*_\lambda} \leqq M_B \| u - v \|_{X_\lambda} ,$$

$$\| Ru - Rv \|_{X_\lambda} \leqq \sqrt{\frac{T}{2\lambda}} \| u - v \|_{X_\lambda} .$$

The assertions of Lemma 5 are proved in [7] (Chap. VII, § 1).

Corollary 2. Let the assumptions of Lemma 5 be satisfied. Then

$$\langle (A + BR)u - (A + BR)v, u - v \rangle_\lambda \geq m_\lambda \| u - v \|^2_{X_\lambda}$$
$$\forall u, \ \forall v \in X$$

and

$$\| (A + BR)u - (A + BR)v \|_{X^*_\lambda} \leqq M_\lambda \| u - v \|_{X_\lambda}$$
$$\forall u, \ \forall v \in X,$$

where $m_\lambda = m - M_B\sqrt{\frac{T}{2\lambda}}$ and $M_\lambda = M + M_B\sqrt{\frac{T}{2\lambda}}$.

<u>Lemma 6</u>. The set $\Lambda \subset X_\lambda \times X_\lambda^*$ defined by

$$\Lambda = \{[v,v'] \mid v \in X_\lambda,\ v' \in X_\lambda^*,\ v(0) = b\}$$

is maximal monotone.

<u>Proof</u>. Let $[v_j,v_j'] \in \Lambda$, $j = 1,2$, and $v = v_1 - v_2$. Then

$$\langle v',v\rangle_\lambda = \int_S e^{-2\ s}(v'(s),v(s))ds$$

$$= \lambda\int_S |v(s)|^2 ds + \frac{1}{2}e^{-2\lambda T}|v(T)|^2 \geq 0.$$

Hence, Λ is monotone. The problem

$$v' + Lv = f,\ v \in X_\lambda,\ v(0) = b,$$

is solvable for every $f \in X_\lambda^*$ (see e.g. Lions-Magenes [9]). Consequently, $\Lambda + L$ is surjective and Λ maximal monotone.

Corollary 2 and Lemma 6 show that we are allowed to apply Theorem 1 to our problem if we choose $\lambda > \frac{M_B^2\ T}{2m^2}$.

<u>Theorem 4</u>. Let $A \in (X \longrightarrow X^*)$ and $B \in (X \longrightarrow X^*)$ be Volterra operators which satisfy the conditions (2) and (28), respectively. Then the problem (27) has a unique solution u for every $a \in V$ and $b \in H$. Let

$\lambda > \frac{M_B^2\ T}{2m^2}$, $m_\lambda = m - M_B\sqrt{\frac{T}{2\lambda}}$, $M_\lambda = M + M_B\sqrt{\frac{T}{2\lambda}}$ and

$0 < r < \frac{2m}{M^2}$. If the sequence (u_i) is determined by

$$
(31) \begin{cases} ru_i'' + Lu_i' = Lu_{i-1}' - r(Au_{i-1}' + Bu_{i-1}), \ u_i' \in X, \\ u_i(0) = a, \ u_i'(0) = b, \ i = 1,2,\dots, \ u_0, u_0' \in X \\ \text{arbitrary}, \end{cases}
$$

then

$$
(\| u_i' - u' \|^2_{X_\lambda} + \| r(u_i'' - u'') \|^2_{X^*_\lambda})^{\frac{1}{2}} \leqq
$$

$$
\leqq \frac{k^i_{r\lambda}}{1 - k_{r\lambda}} \| u_1' - u_0' \|_{X_\lambda} ,
$$

where $k_{r\lambda} = (1 - 2m_\lambda r + M_\lambda^2 r^2)^{\frac{1}{2}} < 1$.

Remark 8. The relation (31) may be replaced by

$rv_i' + Lv_i = Lv_{i-1} - r(A + BR)v_{i-1}, \ v_i \in X,$

$v_i(0) = b, \ i = 1,2,\dots, \ v_0 \in X$ arbitrary.

Thus, at each step of the iteration we have to solve only a first order linear initial value problem. Once v_i is known we obtain $u_i = Rv_i$ by a simple integration.

Remark 9. Combining the iteration method and the Galerkin method one obtains a so called projection-iteration method. In the case of first order evolution equations this method was investigated by Gajewski-Gröger [5], [6]. In the case of the problems considered in Section 2 the convergence of the corresponding projection-iteration methods can be proved nearly in the same way.

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