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SOME FIXED POINT THEOREMS FOR MAPPINGS SATISFYING FRUM - - KETKOV CONDITIONS

R. SCHÖNEBERG, Aachen [1)]

Abstract: The purpose of this paper is to give some new results on fixed points for noncompact mappings in normed linear spaces which behave something like π_1-spaces (e.g. Hilbert-spaces, L^p-spaces ($1 \le p < \infty$)).

Key words: Fixed point theorems; weak and strong Frum-Ketkov contractions; compact and nonexpansive mappings; generalized contractions; Banach-contractions; Rothe-type, Borsuk-type and Lefschetz-type theorems.

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1. Introduction. In this paper we study the existence of fixed points for mappings satisfying so called weak or strong "Frum-Ketkov conditions" (see Definition 2 below). These conditions were introduced in an essential stronger form by R.L. Frum-Ketkov [3] and M.A. Krasnoselskii [5] and subsequently used (in this special form) by R.D. Nussbaum [7],[8], M.A. Krasnoselskii [5], M. Furi and M. Martelli [13] and others in proving fixed point theorems. We establish various existence theorems under certain boundary con-

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1) I would like to thank Prof. J. Reinermann for helpful suggestions.

ditions which include - as special cases - most of the known results of this type and some interesting new ones.

2. <u>Definitions and preliminaries.</u> For a normed linear space (n.l.s.) $(E, \| \|)$, a subset X of E and a map $f: X \longrightarrow E$ we denote by $\overline{X}$, $\partial_E X$ and Fix(f) respectively the closure of X, the boundary of X in E and the fixed point set of f. $A(E, \| \|)$ stands for the collection of all nonempty closed subsets of E.

<u>Definition 1.</u> Let $(E, \| \|)$ be a n.l.s., $\emptyset \neq X \subset E$ and K: $: X \longrightarrow A(E, \| \|)$. K is said to be <u>admissible</u>$:\Longleftrightarrow \overline{\bigcup_{x \in X} K(x)}$

<u>Definition 2.</u> Let $(E, \| \|)$ be a n.l.s., $\emptyset \neq X \subset E$, f: $: X \longrightarrow E$ and let $K: X \longrightarrow A(E, \| \|)$ be admissible.

(i) f is said to satisfy a <u>weak Frum-Ketkov condition with respect to</u> K$:\Longleftrightarrow \bigwedge_{x \in X} d(f(x), K(x)) \leq d(x, K(x))$ 2)

(ii) f is said to satisfy a <u>strong Frum-Ketkov condition with respect to</u> K$:\Longleftrightarrow$

(α) $\bigwedge_{x \in X} d(f(x), K(x)) \leq d(x, K(x))$

(β) $\bigwedge_{(x_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}} [\underline{\lim}(d(x_n, K(x_n)))_{n \in \mathbb{N}} > 0 \Longrightarrow$

$\Longrightarrow \overline{\lim} (d(f(x_n), K(x_n)))_{n \in \mathbb{N}} < \overline{\lim}(d(x_n, K(x_n)))_{n \in \mathbb{N}}]$

<u>Remark 1.</u> If f satisfies a weak (strong) Frum-Ketkov condition with respect to some admissible $K: X \longrightarrow A(E, \| \|)$

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2) For $y \in E$ and $M \subset E$ $d(y, M)$ denotes the distance from y to M

we call f a <u>weak (strong) Frum-Ketkov contraction.</u>

<u>Proposition 1.</u> Let $(E, \| \|)$ be a n.l.s., $\emptyset \neq X \subset E$, $f: X \longrightarrow E$, $M \subset E$ be compact and $m \in [0,1)$ such that

$$\bigwedge_{x \in X} \bigvee_{y \in M} \| f(x) - y \| \leq m \| x - y \|$$

<u>Then</u> f is a strong Frum-Ketkov contraction

<u>Proof:</u> Obvious.

<u>Proposition 2.</u> <u>Let</u> $(E,(,))$ be a Hilbert-space, $\emptyset \neq X \subset E$ and $f: X \longrightarrow E$

<u>Then</u>

(i) If f is a Banach-contraction (i.e. $\bigvee_{\lambda \in [0,1)} \bigwedge_{x,y \in X} \| f(x) - f(y) \| \leq \lambda \| x - y \|$) then f is a strong Frum-Ketkov contraction

(ii) If X is the finite union of closed, convex sets and f is a generalized contraction (i.e. $\bigvee_{\alpha: X \longrightarrow [0,1)} \bigwedge_{x,y \in X} \| f(x) - f(y) \| \leq \alpha(x) \| x - y \|$) then f is a strong Frum-Ketkov contraction

(iii) If X is bounded and f nonexpansive (i.e.) $\bigwedge_{x,y \in X} \| f(x) - f(y) \| \leq \| x - y \|$) then f is a weak Frum-Ketkov contraction

<u>Proof:</u>(i) By a well-known theorem of Kirszbraun and Valentine [12] there is a Banach-contraction $g: E \longrightarrow \overline{co}(f[X])$ 1) such that $g|_X = f$. Choose $y \in Fix(g)$ and define $K: X \longrightarrow A(E,(,))$ by $K(x) := \{y\}$. It is easily seen that f satisfies a strong Frum-Ketkov condition with respect to K.

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1) For $M \subset E$ $\overline{co}(M)$ denotes the convex closure of M

(ii) Let $X = \bigcup_{\nu=1}^{n} C_v$ with C_v nonempty, closed and convex and choose a map $j: X \longrightarrow \{1,\dots,n\}$ with $\bigwedge_{x \in X} x \in C_{j(x)}$. For $v \in \{1,\dots,n\}$ let $r_v: E \longrightarrow C_v$ be a nonexpansive retraction onto C_v. A theorem of W.A. Kirk [4] guarantees that there is for $v \in \{1,\dots,n\}$ exactly one $x_v \in E$ with $f \circ r_v(x_v) = x_v$. If we define $m := \max\{\alpha(r_v(x_v)) \mid v \in \{1,\dots,n\}\}$ we have $m \in [0,1)$ and for $x \in X$: $\|f(x) - x_{j(x)}\| \leq \alpha(r_{j(x)}(x_{j(x)}))\,\|x - x_{j(x)}\| \leq m\,\|x - x_{j(x)}\|$. Hence f is a strong Frum-Ketkov contraction by Proposition 1.

(iii) Analogous to part (i). Q.E.D.

<u>Definition 3.</u> Let $(E, \|\ \|)$ be a n.l.s., $\emptyset \neq X \neq E$ and $f: X \longrightarrow E$. f is said to be <u>demicontinuous</u>: $\Longleftrightarrow$

$$\bigwedge_{(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}} \bigwedge_{x_0 \in X} [\lim (x_n)_{n\in\mathbb{N}} = x_0 \text{ (strongly)} \Longrightarrow$$
$$\Longrightarrow \lim (f(x_n))_{n\in\mathbb{N}} = f(x_0) \text{ (weakly)}$$

3. <u>Fixed points of weak Frum-Ketkov contractions</u>

<u>Lemma 1.</u> <u>Let</u> $(E,(,))$ be a Hilbert-space, $\emptyset \neq C \in H$ and let $P: E \longrightarrow \overline{co}(C)$ be the metric projection onto $\overline{co}(C)$ (i.e. $\bigwedge_{y\in E} \|y - P(y)\| = d(y,\overline{co}(C))$)

<u>Then</u> $\bigwedge_{\emptyset \neq K \subset C} \bigwedge_{y \in E} d(P(y),K)^2 + \|y - P(y)\|^2 \leq d(y,K)^2$

<u>Proof:</u> Let $\emptyset \neq K \subset C$, $y \in E$ and $y_0 \in K$. For $\lambda \in (0,1)$ we have

$$\|y - P(y)\|^2 = d(y,\overline{co}(C))^2 \leq \|y - (\lambda P(y) + (1-\lambda)y_0)\|^2 = \|(y - P(y)) + (1-\lambda)(P(y) - y_0)\|^2 =$$

$= \|y - P(y)\|^2 + (1-\lambda)^2 \|P(y) - y_0\|^2 +$
$+ 2(1-\lambda)\mathrm{Re}(y - P(y), P(y) - y_0)$

hence $0 \leq (1-\lambda)\|P(y) - y_0\|^2 + 2\,\mathrm{Re}(y - P(y), P(y) - y_0)$ and therefore ($\lambda \to 1-$): $0 \leq 2\,\mathrm{Re}(y - P(y), P(y) - y_0)$. This yields

$\|y - P(y)\|^2 + d(P(y),K)^2 \leq \|y - P(y)\|^2 + \|P(y) - y_0\|^2 +$

$+ 2\,\mathrm{Re}(y - P(y), P(y) - y_0) = \|(y - P(y)) + (P(y) - y_0)\|^2 =$

$= \|y - y_0\|^2$

By taking the infimum on the right hand side of the last inequality we get the desired result. Q.E.D.

<u>Theorem 1.</u> <u>Let</u> $(E,(,))$ be a Hilbert-space, $\emptyset \neq X \subset E$ be closed and convex, $K: X \to A(E,(,))$ be admissible such that $(*)$ $\bigwedge_{x \in \partial_E X} K(x) \subset X$ and let the continuous map $f: X \to E$ satisfy a weak Frum-Ketkov condition with respect to K

<u>Then</u> Fix $(f) \neq \emptyset$

<u>Proof:</u> Set $C := \overline{\bigcup_{x \in X} K(x)}$ and let $P: E \to \overline{co}(C)$ and $r: E \to X$ respectively denote the metric projections onto $\overline{co}(C)$ and X. By Schauder's fixed point theorem there is $x \in E$ such that $P \circ f \circ r(x) = x$. From Lemma 1 we have $d(x,K(r(x)))^2 + \|f(r(x)) - x\|^2 \leq d(f(r(x)),K(r(x)))^2 \leq$

$\leq d(r(x),K(r(x)))^2$ and $d(r(x),K(r(x)))^2 + \|x - r(x)\|^2 \leq$

$\leq d(x,K(r(x)))^2$. Combining this, we get $\|x - r(x)\|^2 +$

$+ \|f(r(x)) - x\|^2 \leq 0$ and thus $f(x) = x$. Q.E.D.

Remark 2. (1) M.A. Krasnoselskii's theorem 5 in [5] is a special case of Theorem 1.

(2) It should be noted that we do not assume $f[X] \subset X$ in Theorem 1 and that the assumptions in fact do not even imply $f[\partial_E X] \subset X$ ($E := \mathbb{R}$; $X := [-1,+1]$; $f(x) := -2x$; $K(x) := \{-x\}$)

(3) Corollary 1. Let $(E,(,))$ be a Hilbert-space, $\emptyset \neq X \subset E$ be closed and convex and $f: X \longrightarrow E$ be continuous such that there exists a compact subset M of X with

$$\bigwedge_{x \in X} \bigvee_{y \in M} \|f(x) - y\| \leq \|x - y\|$$

Then $\mathrm{Fix}(f) \neq \emptyset$

4. Fixed points of strong Frum-Ketkov contractions

Lemma 2. Let $(E, \| \ \|)$ be a n.l.s., $\emptyset \neq X \subset E$ be closed, $K: X \longrightarrow A(E, \| \ \|)$ be admissible and let the demicontinuous map $f: X \longrightarrow E$ satisfy a strong Frum-Ketkov condition with respect to K.

Then the following two assertions are equivalent

(i) $\mathrm{Fix}(f) \neq \emptyset$

(ii) For $\varepsilon > 0$ there is a nonexpansive map $P: E \longrightarrow E$ such that $\mathrm{Fix}(P \circ f) \neq \emptyset$ and $\|P(y) - y\| \leq \varepsilon$ for $y \in \bigcup_{x \in X} K(x)$.

Proof: "(i) $\Longrightarrow$ (ii)" For $\varepsilon > 0$ define $P := \mathrm{Id}_E$.

"(ii) $\Longrightarrow$ (i)" Let $C := \overline{\bigcup_{x \in X} K(x)}$. By assumption there are sequences $(P_n)_{n \in \mathbb{N}}$ and $(x_n)_{n \in \mathbb{N}}$ such that

(1) $\bigwedge_{n \in \mathbb{N}} P_n: E \longrightarrow E$ nonexpansive

(2) $\bigwedge_{n \in \mathbb{N}} \bigwedge_{y \in C} \|P_n(y) - y\| \leq \frac{1}{n}$

(3) $\bigwedge_{m \in \mathbb{N}} x_n \in X \wedge P_n \circ f(x_n) = x_n$

If we choose for $n \in \mathbb{N}$ a point $y_n \in K(x_n)$ with $\| f(x_n) - y_n \| = d(f(x_m),K(x_n))$, we get $d(x_n,K(x_n)) \leq$

$$\leq \| x_n - y_n \| \leq \| P_n \circ f(x_n) - P_n(y_n) \| + \| P_n(y_n) - y_n \|$$

$$\leq d(f(x_n),K(x_n)) + \frac{1}{n} .$$

Hence $\overline{\lim}(d(x_n,K(x_n)))_{n \in \mathbb{N}} \leq \overline{\lim}(d(f(x_n),K(x_n)))_{n \in \mathbb{N}}$, which implies: $\underline{\lim}(d(x_n,K(x_n)))_{n \in \mathbb{N}} = 0$. By compactness of C we may assume $\lim(d(x_n,K(x_n)))_{n \in \mathbb{N}} = 0$ and $\lim(x_n)_{n \in \mathbb{N}} = x \in C$, (strongly). The demicontinuity of f yields $\lim(f(x_n))_{n \in \mathbb{N}} = f(x)$ (weakly) and because of $\lim(d(f(x_n),K(x_n)))_n = 0$ we find $\lim(f(x_n))_{n \in \mathbb{N}} = f(x)$ (strongly). The inequality

$$d(f(x),C) \leq d(f(x),K(x_n)) \leq \| f(x) - f(x_n) \| + d(f(x_n),K(x_n))$$

shows $f(x) \in C$. Finally, because

$$\| x - f(x) \| \leq \| x - x_n \| + \| P_n \circ f(x_n) - P_n \circ f(x) \| +$$

$$+ \| P_n \circ f(x) - f(x) \| \leq \| x - x_n \| + \| f(x_n) - f(x) \| + \frac{1}{n}$$

holds for $n \in \mathbb{N}$, we see that x is a fixed point of f.

Q.E.D.

<u>Theorem 2.</u> <u>Let</u> $(E, \| \ \|)$ be a n.l.s., $X \subset E$ be an open, bounded neighborhood of the origin, $K: X \longrightarrow A(E, \| \ \|)$ be admissible and let the demicontinuous map $f: X \longrightarrow E$ satisfy a strong Frum-Ketkov condition with respect to K.
Assume that there exists a sequence of finite-dimensional linear nonexpansive projections $(P_n)_{n \in \mathbb{N}} \in (E^E)^{\mathbb{N}}$ such that

$\lim(P_n(y))_{n\in\mathbb{N}} = y$ (strongly) for $y \in \bigcup_{x\in X} K(x)$. [1)]

Let furthermore one of the following conditions be satisfied:

(a) X is an open ball about the origin and

$$\bigwedge_{x\in\partial_E X} \lim_{\lambda\to 0_+} \left\{\frac{d((1-\lambda)x + \lambda f(x), X)}{\lambda}\right\} = 0$$

(i.e. f is "weakly inward", see [10])

(b) The conjugate space of $(E, \| \|)$ is strictly convex and if $J: E \longrightarrow E^*$ denotes the normalized duality mapping we have

(+) $\bigwedge_{x\in\partial_E X} \operatorname{Re} J(x)(f(x)) \leq \|x\|^2$

or

(++) $\bigwedge_{x\in\partial_E X} \operatorname{Re} J(x)(f(x)) \geq \|x\|^2$

(c) X is an open ball about the origin and $f[\partial_E X] \subset X$

(d) X is symmetric and $\bigwedge_{x\in\partial_E X} f(x) = -f(-x)$

<u>Then</u> $\operatorname{Fix}(f) \neq \emptyset$

<u>Proof:</u> Let $\varepsilon > 0$. By a standard argument there is $n \in \mathbb{N}$ such that $\|P_n(y) - y\| \leq \varepsilon$ for $y \in \bigcup_{x\in X} K(x)$. In view of Lemma 2 it remains to show that each of the conditions (a) - (d) implies $\operatorname{Fix}(P_n \circ f) \neq \emptyset$. Let $H := P_n[E]$.

(a) Using $P_n[\overline{X}] = \overline{X} \cap H$ it is easily seen that $P_n \circ f$ is weakly inward, too, when restricted to $\overline{X} \cap H$. By a well-

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1) Such a sequence exists f.e. if $(E, \| \|)$ is a $(\pi)_1$-space (see [8])

known result there is $x \in \overline{X} \cap H$ with $P_n \circ f(x) = x$.

(b) From [2] we learn that $\bigwedge_{y \in H} J(y) \circ P_n = J(y)$. Hence $P_n \circ f|_{\overline{X} \cap H}$ satisfies (+) or (++) (with X replaced by $X \cap H$) according as (+) or (++) holds for f. Therefore $\mathrm{Fix}(P_n \circ f) \neq \emptyset$ by the Leray-Schauder theorem for finite-dimensional spaces.

(c) We have (a) fulfilled and therefore $\mathrm{Fix}(P_n \circ f) \neq \emptyset$.

(d) The antipodal-theorem for finite-dimensional spaces yields the existence of $x \in H \cap \overline{X}$ with $P_n \circ f(x) = x$. Q.E.D.

Remark 3. (1) Theorem 2 with condition (c) improves a result of R.D. Nussbaum [8] where f is assumed to be a continuous map such that

$$\bigvee_{\emptyset \neq M \subset E \text{ compact}} \bigvee_{k \in [0,1)} \bigwedge_{x \in X} d(f(x),M) \leq k d(x,M)$$

(2) Corollary 2. Let $(E,(,))$ be a Hilbert-space, $X \subset E$ be an open, bounded neighborhood of the origin and let $f: X \longrightarrow E$ be a demicontinuous strong Frum-Ketkov contraction such that

$$(\cdot) \quad \bigwedge_{x \in \partial_E X} \mathrm{Re}\ (f(x),x) \leq \|x\|^2$$

or

$$(\cdot\cdot) \quad \bigwedge_{x \in \partial_E X} \mathrm{Re}\ (f(x),x) \geq \|x\|^2 .$$

Then $\mathrm{Fix}(f) \neq \emptyset$

(3) Corollary 2 improves the results of M.A. Krasnoselskii and P.P. Zabreiko [6] and J. Reinermann [9].

(4) Conditions ($\cdot$) and ($\cdot\cdot$) of Corollary 2 are res-

pectively equivalent to

$$\text{(I)} \quad \bigwedge_{x \in \partial_E X} \| f(x) - x \|^2 \geq \| f(x) \|^2 - \| x \|^2$$

and

$$\text{(II)} \quad \bigwedge_{x \in \partial_E X} \| f(x) - x \|^2 \leq \| f(x) \|^2 - \| x \|^2$$

(5) A long but elementary computation yields that if X is an open ball about the origin (•) of Corollary 2 is equivalent to (• • •) f is weakly inward

We end this paper in proving a Rothe-type theorem. We will need the following two lemmata. The first is a well-known result and the second is proved in a more general form in [10]. For the sake of completeness we give the proof of the second one.

<u>Lemma 3.</u> <u>Let</u> $(E,(,))$ be a Hilbert-space, $n \in \mathbb{N}$, $x_1,\dots,x_n \in E$, $M \subset E$ be compact and $\varepsilon > 0$

<u>Then</u> there is a finite-dimensional subspace H of E such that $\{x_1,\dots,x_n\} \subset H$ and the orthogonal projection $P: E \longrightarrow$ $\longrightarrow H$ satisfies $\| P(y) - y \| \leq \varepsilon$ for $y \in M$

<u>Lemma 4.</u> <u>Let</u> $(E, \| \ \|)$ be a n.l.s. and $\emptyset \neq X \subset E$ be a finite union of closed convex subsets of E such that X is contractible. Let $f: X \longrightarrow E$ be compact with $f[\partial_E X] \subset X$

<u>Then</u> $\mathrm{Fix}(f) \neq \emptyset$

<u>Proof:</u> Since X is an ANR (see [1]), a well-known result of Borsuk implies (by contractibility of X) that X is an AR. If $r: E \longrightarrow X$ denotes a retraction onto X, we define $g: E \longrightarrow E$ by $g(x) := \begin{cases} f(x) & x \in X \\ r \circ f \circ r(x) & x \in E \setminus X \end{cases}$.

Since g is compact, Schauder's fixed point theorem yields $Fix(f) \neq \emptyset$. Because of $Fix(f) = Fix(g)$ we are done. Q.E.D.

Theorem 3. Let $(E,(,))$ be a Hilbert-space and $X \subset E$ be finite union of closed balls such that X is contractible in the weak or strong topology of $(E,(,))$. Let $f: X \longrightarrow E$ be a demicontinuous strong Frum-Ketkov contraction satisfying $f[\partial_E X] \subset X$

Then $Fix(f) \neq \emptyset$

Proof: Choose an admissible $K: X \longrightarrow A(E,(,))$ such that f satisfies a strong Frum-Ketkov condition with respect to K. Set $C := \overline{\bigcup_{x \in X} K(x)}$ and let $\varepsilon > 0$. By assumption there are $n \in \mathbb{N}$, $x_1,\dots,x_n \in E$ and $r_1,\dots,r_n \in (0,\infty)$ with $X = \bigcup_{\nu=1}^{n} \overline{B}(x_\nu, r_\nu)$. By Lemma 3 there is a finite-dimensional subspace H of E such that $\{x_1,\dots,x_n\} \subset H$ and the orthogonal projection $P: E \longrightarrow H$ satisfies $\|P(y) - y\| \leq \varepsilon$ for $y \in C$. We have $P[X] = X \cap H$ and thus $X \cap H$ is the finite union of compact convex sets and contractible. Because $P \circ f[\partial_H(X \cap H)] \subset P \circ f[\partial_E X \cap H] \subset P[X] \subset X \cap H$ we have $Fix(P \circ f) \neq \emptyset$ by Lemma 4. Lemma 2 gives the conclusion. Q.E.D.

Remark 4. (1) We do not know, whether the assumption "X is the finite union of closed balls" can be weakened to "X is the finite union of closed convex sets". In this context it should be noted that M. Furi and M. Martelli [13], in extending a result of R.D. Nussbaum [7], recently proved that (strongly) contractible subsets of arbitrary Banach spaces, which are finite unions of closed, bounded and convex sets, have the fixed point property for special selfmappings of Frum-Ketkov type, namely those we described in Remark 3 (1). Unfortunately their argument doesn't work in the general setting, although it seems to be very useful in the area of fixed point theory (see [10]).

(2) Proposition 2 (ii) and Theorem 3 gives a fixed point result for generalized contractions, which is extended to nonexpansive mappings in [10] .

NOTE ADDED IN PROOF.

J. Daneš from the Charles University, Prague, has indicated to the author that Lemma 1 can be found in the appendix to "Topological Methods in Nonlinear Analysis" (Charles University, 1972/73) written by J. Kolomý and J. Daneš.

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