Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 2, 345--355

Persistent URL: http://dml.cz/dmlcz/105699

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,2 (1976)

A NOTE ON TEST MODULES

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Abstract: Sometimes, it is useful to have a criterion to determine whether a module is injective, simply by testing its injectivity with respect to submodules of a fixed module. This problem has been studied by several authors, e.g. the well-known Baer's criterion states that every ring R is a test module for injectivity in the category of R-modules. In this paper, several characterizations of test modules for injectivity are presented. Further, an attempt is made to dualize some of these results.

Key words: Injective module, projective module, test module, centrally splitting preradical.

AMS: 16A52 Ref. Ž.: 2.723.2

By R-mod we understand the category of unital left modules over an associative ring R with unit. First, several basic facts concerning preradicals, which are going to be our main tool. A preradical r for R-mod is a subfunctor of the identity functor, i.e. r assigns to each module M its submodule r(M) in such a way that every homomorphism of M into N induces a homomorphism of r(M) into r(N) by restriction. For every preradical r we define the class of r-torsion modules by $\mathcal{T}_r = \{M \in R\text{-mod} \mid r(M) = M\}$ and the class of r-torsionfree modules by $\mathcal{F}_r = \{M \in R\text{-mod} \mid r(M) = 0\}$. A module M splits in r if r(M) is a direct summand of M. We shall say that a preradical r is

- idempotent, if $r(r(M)) = r(M)$ for all $M \in R$-mod,
- a radical if $r(M/r(M)) = 0$ for all $M \in R$-mod,
- hereditary if $r(N) = N \cap r(M)$ for all $n \subseteq M$, $M \in R$-mod,
- cohereditary if $r(M/N) = r(M) + N/N$ for all $N \subseteq M$, $M \in R$-mod,
- stable if every injective module splits in r,
- costable if R and consequently every projective module splits in r,
- splitting if every module splits in r,
- centrally splitting if $r(R)$ is a ring direct summand in R and r is cohereditary.

With every preradical r we associate preradicals $h(r)$ and $ch(r)$ defined by $h(r)(M) = M \cap r(E(M))$, where $E(M)$ denotes the injective hull of M, and $ch(r)(M) = r(R)M$. Obviously, $h(r)$ is hereditary and $ch(r)$ is cohereditary. For every module M we define preradicals p_M and p^M by $p_M(Q) = \sum \operatorname{Im} f$, $f \in \operatorname{Hom}(M,Q)$, and $p^M(Q) = \bigcap \operatorname{Ker} f$, $f \in \operatorname{Hom}(Q,M)$, for all $Q \in R$-mod. Finally, we shall say that $0 \to K \to P \to M \to 0$ is a projective cover of M if P is projective and K is small in P, i.e. $K + N = P$ implies $N = P$.

We shall need the following simple result.

<u>Lemma 1</u>: Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \longrightarrow & C & \xrightarrow{p} & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & X & \xrightarrow{j} & Y & \longrightarrow & Z & \xrightarrow{q} & 0 \end{array}$$

be a commutative diagram with exact rows and $\varphi : B \to X$, $\psi : C \to Y$ be such that $\varphi j + p \psi = g$. Then

(i) if Ker p = Ker g.q and Im j is essential in Y then Im i is essential in B and $\varphi j = g$,

(ii) if Im j = Im ig and Im i is small in B then Im j is small in Y and $p\psi = g$.

Proof: (i) Obviously, Ker p = Ker g.q means nothing else than Im i = g^{-1}(Im j) and hence Im i is essential in B. Let $y \in \mathrm{Im}\, j \cap \mathrm{Im}\, (g - \varphi j)$. Then there are $x \in X$, $b \in B$ with $xj = y = bg - b\varphi j$, hence $bg = (x + b\varphi) j \in \mathrm{Im}\, j$, and so $b = ai$ for some $a \in A$. Now we have $y = b(g - \varphi j) = aip\psi = 0$.

(ii) It is easy to see that Im j is small in Y. Further, for each $b \in B$ there is $a \in A$ with $b\varphi j = aig = ai(\varphi j + p\psi) = ai\varphi j$. Then, however, $b - ai \in \mathrm{Ker}\, \varphi j = \mathrm{Ker}\, (g - p\psi)$, so that $B = \mathrm{Ker}\, (g - p\psi) + \mathrm{Im}\, i$.

Now we present several results concerning M-injectivity. These results are already known, however our proofs are very easy. In particular, we get an extremely simple characterization of M-injective hulls. Let M, $Q \in$ R-mod. Recall that Q is said to be M-injective if every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & M \\ & & {\scriptstyle f}\downarrow & & \\ & & Q & & \end{array}$$

with exact row can be completed.

<u>Proposition 2</u>: Let M, $Q \in$ R-mod. The following conditions are equivalent:

(i) Q is M-injective,

(ii) every diagram $\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i} & M \\ & & \downarrow & & \\ & & Q & & \end{array}$ with exact row and

Im i essential in M, can be completed,

(iii) Im $f \subseteq Q$ for every $f \in \text{Hom}(M,E(Q))$,

(iv) $p_M(E(Q)) \subseteq Q$.

Proof: The implications (i) implies (ii) and (iii) implies (iv) are obvious, while the implication (ii) implies (iii) follows immediately from Lemma 1 (i).

(iv) implies (i). Let $A \subseteq M$ and $f \in \text{Hom}\,(A,Q)$. There is $g \in \text{Hom}\,(M,E(Q))$ making the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & M \\ & & f\downarrow & \swarrow g & \\ & & Q & & \\ & & \downarrow & & \\ & & E(Q) & & \end{array}$$

commutative. However, Im $g \subseteq p_M(E(Q)) \subseteq Q$ and we are through.

Proposition 3: Let M, $Q \in$ R-mod and $r = p_M$. The following are equivalent:

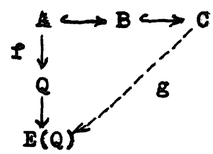
(i) Q is M-injective,

(ii) every diagram $\begin{array}{ccc} A & \hookrightarrow & B \\ f\downarrow & & \\ Q & & \end{array}$ such that there is $C \in$ R-mod with $B \subseteq C$ and C/Ker $f \in \mathcal{T}_r$ can be completed,

(iii) every diagram $\begin{array}{ccc} A & \hookrightarrow & B \\ f\downarrow & & \\ Q & & \end{array}$ with B/Ker $f \in \mathcal{T}_{h(r)}$ can be completed,

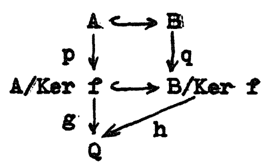
(iv) every diagram $\begin{array}{ccc} I & \hookrightarrow & R \\ f\downarrow & & \\ Q & & \end{array}$ with R/Ker $f \in \mathcal{T}_{h(r)}$ can be completed.

Proof: (i) implies (ii): Consider the commutative diagram



where $C/\ker f \in \mathcal{T}_r$. Since $\operatorname{Ker} f \subseteq \operatorname{Ker} g$ and $\operatorname{Im} g \cong C/\operatorname{Ker} g$, we have $\operatorname{Im} g \in \mathcal{T}_r$ and Proposition 2 (iv) yields $\operatorname{Im} g \subseteq Q$.

(ii) implies (iii). Consider the commutative diagram



where p, q are natural epimorphisms, g is a monomorphism and $pg = f$. Since $B/\operatorname{Ker} f \in \mathcal{T}_{h(r)}$, $B/\operatorname{Ker} f \subseteq r(E(B/\operatorname{Ker} f)) \in \mathcal{T}_r$ and, by (ii), there is $h: B/\operatorname{Ker} f \longrightarrow Q$ making the whole diagram commutative.

(iii) implies (iv) obviously.

(iv) implies (i). Let $A \subsetneqq M$, $x \in M \setminus A$, $f: A \longrightarrow Q$ be such that f cannot be extended to a larger submodule of M. Put $I = (A:x)$, and define $g: I \longrightarrow Q$ by $rg = rxf$ for all $r \in I$. Denote $K = \operatorname{Ker} g$ and $L = \operatorname{Ker} f$. Then $K = (L:x)$ and $R/K \cong (Rx + L)/L \in \mathcal{T}_{h(r)}$. Hence g can be extended to $h: R \longrightarrow Q$ and we can define $k: Rx + A \longrightarrow Q$ by $(rx + a)k = r(1h) + af$ for all $a \in A$, $r \in R$, a contradiction.

<u>Proposition 4:</u> Let M, $Q \in R$-mod and $Q_M = Q + p_M(E)Q))$. Then

(i) Q_M is M-injective,

(ii) if $Q \subseteq N$ and N is M-injective then there is a monomorphism $f: Q_M \longrightarrow N$ such that $f \mid Q = 1_Q$.

Proof: Since $Q \subseteq Q_M \subseteq E(Q)$, $p_M(E(Q)) = p_M(E(Q)) \subseteq Q_M$ and Q_M is M-injective by Proposition 2 (iv). If $Q \subseteq N$ for some M-injective module N, we have a monomorphism $g: E(Q) \longrightarrow E(N)$ with $g \mid Q = 1_Q$. However, $p_M(E(Q))g \subseteq p_M(E(N)) \subseteq N$, so $f = g \mid Q_M$ has the desired property.

Now we turn our attention to test modules. A module M is said to be a test module for injectivity if every M-injective module is injective.

Proposition 5: Let $M \in$ R-mod and $r = p_M$. The following are equivalent:

(i) M is a test module for injectivity,

(ii) $E(Q) = Q + r(E(Q))$ for all $Q \in$ R-mod,

(iii) If $Q \in$ R-mod and every homomorphism $f: I \longrightarrow Q$, where I is a left ideal and $R/\mathrm{Ker}\, f \in \mathcal{T}_{h(r)}$, can be extended to $g: R \longrightarrow Q$, then Q is injective,

(iv) if $Q \in$ R-mod and every diagram $\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i} & M \\ & & f\downarrow & & \\ & & Q & & \end{array}$ with exact row and Im i essential in M can be completed then Q is injective.

Proof: This is an immediate consequence of Propositions 2, 3, 4.

Theorem 6: Let $M \in$ R-mod and $r = p_M$. The following are equivalent:

(i) M is a test module for injectivity,

(ii) h(r) is centrally splitting and every h(r)-torsionfree module is completely reducible,

(iii) $I = h(r)(R)$ is a ring direct summand in R and R/I is a completely reducible ring.

Proof: (i) implies (ii). For every $N \in \mathcal{T}_r$, $E(N) = N + r(E(N)) = r(E(N))$, and hence r is stable by [1, Proposition 2.4] . Further, if $Q \in \mathcal{T}_{h(r)}$, then Hom $(M,E(Q)) = 0$, and sc Q is M-injective by Proposition 2 (iii). Thus every h(r)-torsionfree module is injective, and consequently completely reducible (since $\mathcal{F}_{h(r)}$ is closed under submodules). In particular, $\mathcal{F}_{h(r)}$ is closed under factor-modules. Since r is stable, h(r) is so by [1, Theorem 2.6] and therefore h(r) is a radical by [1, Proposition 2.5] . Moreover, h(r) is cohereditary by [3, Proposition 4.1] . However, every stable hereditary cohereditary radical is centrally splitting by [2, Proposition 5] .

(ii) implies (iii) trivially.

(iii) implies (i). For each module Q we have the canonical decomposition $E(Q) = A \oplus B$, where $A = IE(Q)$ and B is completely reducible. If Q is M-injective then $IE(Q) \subseteq r(E(Q)) \subseteq Q$, and so $Q = A \oplus (B \cap Q)$. However, both A and $B \cap Q$ are injective.

<u>Proposition 7</u>: Let $M \in$ R-mod and $r = p_M$. The following are equivalent:

(i) E(R) is a homomorphic image of a direct sum of copies of M,

(ii) M is a faithful test module for injectivity,

(iii) $h(r)(N) = N$ for all $N \in$ R-mod,

(iv) every injective module is r-torsion.

Proof: (i) implies (ii). We have $E(R) = r(E(R)) =$

= h(r)(E(R)), so h(r)(R) = R and M is a test module for injectivity by Theorem 6 (iii), Further, aM = 0 yields aE(R) = = 0, and hence a = 0.

(ii) implies (iii). Put I = h(r)(R). By Theorem 6, I is a ring direct summand of R, R = I ⊕ K. However, h(r) is cohereditary, hence M = h(r)(M) = IM and KM = KIM = 0 yields K = = 0, M being faithful.

(iii) implies (iv) and (iv) implies (i) trivially.

<u>Corollary 8</u>: A module M is a generator for R-mod iff M is a faithful test module for injectivity and p_M is hereditary.

In the final part we make an attempt to dualize some of our results. After giving a characterization of M-projective modules with projective covers, we shall proceed immediately to the dualization of Theorem 6. In order to get a complete dualization of Theorem 6, we must restrict ourselves to the case of left perfect rings. This restriction plays a serious rôle here, as the recent solution of Whitehead's problem (see [4]) seems to indicate.

Let M ∈ R-mod. Recall that a module Q is said to be M-projective if every diagram in the form

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow f & & \\ M & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact row can be completed. We shall say that M is a test module for projectivity if every M-projective module is projective.

<u>Proposition 9</u>: Let M, $Q \in R$-mod and $0 \to K \to P \to Q \to 0$ be a projective cover of Q. The following are equivalent:

(i) Q is M-projective,

(ii) every diagram

$$\begin{array}{ccccc} & & Q & & \\ & & \downarrow & & \\ M & \xrightarrow{p} & N & \longrightarrow & 0 \end{array}$$

with ker p small in M can be completed,

(iii) $K \subseteq \operatorname{Ker} f$ for every $f \in \operatorname{Hom}(P,M)$,

(iv) $K \subseteq p^{M}(P)$.

Proof: (i) implies (ii) and (iii) implies (iv) trivially while (ii) implies (iii) by Lemma 1 (ii).

(iv) implies (i). Considering the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \xrightarrow{p} & Q & \longrightarrow & 0 \\ & & & & g\downarrow & & \downarrow f & & \\ & & & & M & \xrightarrow{q} & N & \longrightarrow & 0 \end{array}$$

we have $K \subseteq p^{M}(P) \subseteq \operatorname{Ker} g$. Hence there is $h: Q \to M$ with $ph = g$, and consequently $hq = f$.

<u>Theorem 10:</u> Let $M \in R$-mod and $r = p^{M}$. Consider the following conditions:

(i) M is a test module for projectivity,

(ii) ch(r) is centrally splitting and every ch(r)-torsion module is completely reducible,

(iii) $I = (0:M) = r(R)$ is a ring direct summand of R and it is a completely reducible ring,

(iv) every M-projective module possessing projective cover is projective.

Then (ii) and (iii) are equivalent, (i) implies (ii) and (iii) implies (iv). Moreover, if R is left perfect then all these conditions are equivalent.

Proof: The equivalence of (ii) and (iii) is easily seen. Moreover, if R is left perfect then (iv) obviously implies (i).

(i) implies (ii). Let I = r(R). Since M is an R/I module and R/I is a free R/I-module, R/I is M-projective as an R/I-module, and consequently as an R-module. Hence R/I is projective and I is a left direct summand. Therefore ch(r) is costable by [1, Theorem 3.8] and hence idempotent by [1, Proposition 3.5]. Further, if IQ = Q for some Q R-mod, then Hom (Q,M/N) = 0 for all $N \subseteq M$, and so Q is M-projective, thus being projective. Consequently, every ch(r)-torsion module is completely reducible (since $\mathcal{T}_{ch(r)}$ is closed under factor-modules) and, in particular, $\mathcal{T}_{ch(r)}$ is closed under submodules. Thus ch(r) is costable, hereditary and cohereditary, which means that ch(r) is centrally splitting by [2, Proposition 5].

(iii) implies (iv). Let Q be M-projective and $0 \longrightarrow K \longrightarrow P \longrightarrow Q \longrightarrow 0$ be a projective cover of Q. We have $P = IP \oplus A$ and, with respect to Proposition 9 (iv), $K \subseteq r(P)$ and r(P) = IP, P being projective. Thus K is a direct summand in IP, IP being completely reducible, and so $Q = IP/K \oplus A$ is projective.

<u>Corollary 11</u>: Let R be a left perfect ring. Then every faithful module is a test module for projectivity.

R e f e r e n c e s

[1] L. BICAN, P. JAMBOR, T. KEPKA, P. NĚMEC: Stable and co-stable preradicals, Acta Univ. Carolinae, Math. et Phys. 16(1975),63-69.

[2] L. BICAN, P. JAMBOR, T. KEPKA, P. NĚMEC: Centrally splitting preradicals Comment. Math. Univ. Carolinae 17(1976), 31-47.

[3] L. BICAN, P. JAMBOR, T. KEPKA, P. NĚMEC: Hereditary and cohereditary preradicals (to appear in Czech. Math. J.).

[4] S. SHELAH: Infinite abelian groups, Whitehead problem and some constructions, Israel J. Math. 18(1974), 243-256.

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(Oblatum 24.11.1975)