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ON THE EXISTENCE OF WEAK SOLUTIONS FOR SOME QUASILINEAR ELLIPTIC VARIATIONAL BOUNDARY VALUE PROBLEMS AT RESONANCE

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Abstract: The equation Au = Bu under variational boundary conditions in the sense of F.E. Browder is considered, where A is a symmetric, uniformly strongly elliptic, linear partial differential operator with nonzero null space, and B is a sublinear one of the same order with a suitable asymptotic behavior with respect to the null space of A. If B satisfies a Lipschitz condition for the terms of highest order, the existence of a weak solution is proved. Properties of selfadjoint Fredholm operators in regard to the set-measure of noncompactness and set-contractions are the basic tools of this paper.

Key words: Coincidence degree, set-measure of noncompactness, set-contractions, Fredholm operators, alternative problem, boundary value problem, nonlinear partial differential equations.

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Introduction: Let Ω be a bounded domain in $\mathbb{R}^N (N \in \mathbb{N})$, A be a linear, uniformly strongly elliptic, symmetric partial differential operator on Ω of order $2m (m \in \mathbb{N})$, and B be a sublinear partial differential operator of order $\tilde{m} (\tilde{m} \leq 2m)$, given in divergence form. The following destination is often drawn: Au = Bu is called a quasilinear equation, if $\tilde{m} = 2m$, a semilinear, if $\tilde{m} < 2m$.

In this paper we are concerned with variational boundary value problems, in the sense of F.E. Browder ([1]), for Au = = Bu in the quasilinear case, when Au = 0 has at least one nonzero solution, satisfying the boundary conditions (i.e. the resonance case).

The study of such problems for the semilinear equation was initiated by Landesman and Lazer in 1970 ([10]) and is continued by De Figueiredo, Fučík, Hess, Kučera, Mawhin, Nečas, Nirenberg, Schechter and Williams (see e.g.: [2],[3], [4],[5],[11],[13],[14],[15],[17]). In order to ensure the existence of solutions in that case, they use the Hilbert space approach and "topological" arguments, like Schauder's fixed point theorem or the degree theory for completely continuous nonlinearities.

Even, if a Sobolev-Rellich embedding theorem is applicable, the nonlinear part is no longer completely continuous in the quasilinear case. But, when B satisfies a Lipschitz condition with respect to the derivatives of order 2m, we can use a coincidence degree continuation theorem for nonlinearitires, which are set-contractions, for deriving the operator theoretic results, we need. This theorem is stated in [7] and derived in a more general version in [6].

Section 1 contains the later needed notations and assertions. In Section 2 we compute the lower bound of a linear selfadjoint Fredholm operator in a Hilbert space with respect to the set-measure of noncompactness by its essential spectrum, which is basic for our existence theorem. In Section 3 the boundary value problem is formulated and solved. Special cases are mentioned in Section 4.

1. Here we recall some definitions and preliminary results. Let Z be a metric space and M be a subset of Z, then the set-measure of noncompactness γ of M is defined by:

$\gamma(M) := \inf \{\varepsilon \mid \varepsilon > 0$, there is a finite covering of M by subsets of Z with diameter lower than $\varepsilon\}$.

For metric spaces, Z, $\tilde{Z}$ and $k \in \mathbb{R}^+$ we call a continuous function $f\colon Z \longrightarrow \tilde{Z}$ a k-set-contraction, iff $\gamma(f(M)) = k\gamma(M)$ for each bounded subset M of Z, and completely continuous, iff $\overline{f(M)}$ is compact for each bounded $M \subseteq Z$. Further for a function f D(f) denotes the domain of f, R(f) the range. Concerning the coefficient field of the here considered spaces, we suppose $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ in general, but $\mathbb{K} = \mathbb{R}$, if a real valued function space occurs.

Let X, Y be Banach spaces and $L\colon X \supseteq D(L) \longrightarrow Y$ be a closed linear operator, then L is said to be a $\emptyset_+$-operator, iff the null space of L, denoted by Ker(L), is finite dimensional, and R(L) is closed. If additionally $Y \mid R(L)$ is finite dimensional, we call L a Fredholm operator and $\operatorname{ind}(L) := \dim(Y \mid R(L))$ the Fredholm index of L. Further we set:

$\ell(L) := \sup \{r \mid r \in \mathbb{R}^+, r\gamma(M) \leq \gamma(L(M))$ for each bounded $M \subseteq D(L)\}$.

In [6] it is shown that $\ell(L) > 0$, iff L is a $\emptyset_+$-operator.
Now let us make the following assumptions:

(a) X, Y are Banach spaces and $L\colon X \supseteq D(L) \longrightarrow Y$ is a Fredholm operator with $\operatorname{ind}(L) = 0$

(b) $k \in [0, \ell(L))$ and $N\colon X \longrightarrow Y$ is a k-set-contraction.

(a) involves the existence of continuous projectors $P\colon X \longrightarrow X$

and $Q: Y \longrightarrow Y$ with $R(P) = Ker(L)$ and $Ker(Q) = R(L)$, and of a linear isomorphism $J: R(Q) \longrightarrow Ker(L)$. Further we denote the pseudo-inverse of L associated to P by K_P, i.e. $K_P :=$ $:= (L \mid (I - P)(X))^{-1}$. The following assertion is basic in regard to Section 3.

<u>Theorem 1</u>: Let (a) and (b) be satisfied and P, Q, K_P and J be defined like above. Assume further:

(1) There are $\sigma \in [0,1)$ and ϑ, $\mu \in \mathbb{R}^+$, such that for $x \in X$:

$$\| K_P \circ (I - Q) \circ Nx \| \leq \mu \| x \|^{\sigma} + \vartheta .$$

(2) For each bounded subset W of $R(I - P)$ there exists a $t_0 > 0$, such that for all $t \geq t_0$, all $z \in W$, and all $w \in Ker(L)$ with $\| w \| = 1$ we have: $Q \circ N(tw + t^{\sigma} z) \neq 0$.

(3) There is a $t_0 > 0$ with:
$\deg(J \circ Q \circ N \mid Ker(L), \{x \mid x \in Ker(L), \| x \| < t\}, 0) \neq 0$ for $t \geq t_0$.

Then $R(L - N) \supseteq R(L)$.

Here deg means the degree for a finite dimensional normed space. The proof is straightforward in regard to the proof of Theorem VI.4 in [11] and the degree continuation result for k-set-contraction in [6].

2. In dealing with applications, a calculation of $\mathcal{L}(L)$ for a given Fredholm operator L is necessary. Direct estimation can be given in the case of ordinary differential equations ([7],[8],[9]), but they fail, treating partial differential equations. Another computation, developed in this section, is available however, if the given problem involves

a selfadjoint Fredholm operator in a Hilbert space.

Let H be a Hilbert space over $\mathbb{K}$ and L: $H \supseteq D(L) \longrightarrow H$ be a closed linear operator, then we denote the spectrum of L by $\sigma(L)$ and define the essential spectrum of L by:

$\sigma_e(L) := \{ \lambda \mid \lambda \in \sigma(L),\ \lambda$ is not an isolated eigenvalue of finite multiplicity $\}$.

Observe that many different definitions are used, but that they all coincide, when L is selfadjoint. Now we can prove:

<u>Theorem 2:</u> Let H be a Hilbert space over $\mathbb{K}$ and L: $: H \supseteq D(L) \longrightarrow H$ be a closed, selfadjoint, linear operator.

Then $\ell(L) = \inf \{ |\lambda| \mid \lambda \in \sigma_e(L) \}$.

<u>Proof</u>. For convenience we set: $Q := \inf \{|\lambda| \mid \lambda \in \sigma_e(L)\}$.

1) We show: $\ell(L) \leq Q$.

It is well-known that for $\lambda \in \sigma_e(L)$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \in D(L)^{\mathbb{N}}$ with: $\|x_n\| = 1$ for $n \in \mathbb{N}$, $\lim(\lambda x_n - Lx_n)_{n\in\mathbb{N}} = 0$, and $(x_n)_{n\in\mathbb{N}}$ has no convergent subsequence. Hence: $\gamma(\{x_n \mid n \in \mathbb{N}\}) > 0$ and $\gamma(\{Lx_n \mid n \in \mathbb{N}\}) = \gamma(\{\lambda x_n \mid n \in \mathbb{N}\})$. Then $\gamma(\{\lambda x_n \mid n \in \mathbb{N}\}) = |\lambda|\, \gamma(\{x_n \mid n \in \mathbb{N}\})$ involves: $\ell(L) \leq |\lambda|$ for each $\lambda \in \sigma_e(L)$, therefore $\ell(L) \leq Q$.

2) We show: $\ell(L) \geq Q$.

If $0 \in \sigma_e(L)$, the assertion is obvious. Otherwise $Q > 0$ and L is a Fredholm operator with $\mathrm{ind}(L) = 0$, because L is assumed to be selfadjoint. We first consider the case $\mathbb{K} = \mathbb{C}$. Since L is selfadjoint, L is reduced by Ker(L) and $\mathrm{Ker}(L)^{\perp}$. We set $L_1 := L \mid \mathrm{Ker}(L)^{\perp}$ and note that L_1 is an injective, selfadjoint, linear operator in $\mathrm{Ker}(L)^{\perp}$. It follows from $\sigma(L) = \sigma(L\ \mathrm{Ker}(L)) \cup \sigma(L_1)$ and $\sigma(L \mid \mathrm{Ker}(L)) = \{0\}$ that

$\sigma_e(L) = \sigma_e(L_1)$. Since $Q > 0$ Proposition 1.1 (b) in [16] says that L_1^{-1} is a k-set-**contraction** with $k \leq Q^{-1}$. We show $\ell(L_1) \geq \geq Q$. Let $B \subseteq D(L_1)$ be bounded. We can assume $\gamma(L_1(B)) < \infty$, and obtain from $\gamma(L_1^{-1}(L(B))) \leq k\,\gamma(L(B))$: $k^{-1}\gamma(B) \leq \gamma(L(B))$, which implies $k^{-1} \leq \ell(L_1)$, hence $\ell(L_1) \geq Q$. If P is the orthogonal projector on Ker(L), we have for each bounded subset B of D(L):

$$\gamma(B) = \gamma[(P + I - P)(B)] \leq \gamma[P(B) + (I - P)(B)] \leq$$
$$\leq \gamma(P(B)) + \gamma((I - P)(B)) = \gamma((I - P)(B)) \leq \gamma(B),$$

using that I - P is nonexpansive and P is completely continuous. Hence $\gamma(B) = \gamma((I - P)(B))$.

Since Ker(L) and $\text{Ker}(L)^{\perp}$ reduce L, $L \circ (I - P)$ is equal to $(I - P) \circ L$ and we conclude analogously:

$$\gamma(L(B)) = \gamma[L \circ (P + I - P)(B)] \leq \gamma(L \circ (I - P)(B))$$
$$= \gamma((I - P) \circ L(B)) = \gamma(L(B)),$$

which verifies $\gamma(L(B)) = \gamma(L_1 \circ (I - P)(B))$. Both assertions together ensure $\ell(L) = \ell(L_1)$, which proves 2) in the complex case.

For $\mathbb{K} = \mathbb{R}$ we consider the complexification H^+ of H and the operator L^+, induced by L. L^+ is selfadjoint and this implies: $\sigma_e(L) = \sigma_e(L^+)$. Therefore $\ell(L^+) \geq Q$. On the other hand we obtain for $\varepsilon > 0$ and a bounded $B \subseteq D(L)$:

$$\gamma(L(B)) = \gamma(L^+(B \times \{0\})) \geq (\ell(L^+) - \varepsilon)\,\gamma(B \times \{0\}) =$$
$$= (Q - \varepsilon)\,\gamma(B),$$

which involves: $\ell(L) \geq Q$.

3. Now we can treat the boundary value problem, which is indicated in the introduction. First of all some notations

and conventions. We always consider real-valued functions, defined on a bounded domain Ω of the $\mathbb{R}^N$ with $N \in \mathbb{N}$. For $\alpha \in Z^{+N}$ we set $|\alpha| = \sum_{1 \le i \le N} pr_i(\alpha)$, where pr_i means the i-th coordinate projection, and denote the α-th derivative in the weak sense for a function u on Ω by $D^\alpha u$. For $m \in Z^+$ the Sobolev space $W^{m,2}(\Omega)$ is defined by:

$$W^{m,2}(\Omega) := \{u \mid u \in L^2(\Omega),\ D^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \le m\}.$$

For $u, v \in W^{m,2}(\Omega)$ an inner product is given by:

$$\langle u,v \rangle_{m,2} := \sum_{|\alpha| \le m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx.$$

The norm, associated to $\langle\ ,\ \rangle_{m,2}$, will be denoted by $\|\ \|_{m,2}$. Let $C_0^\infty(\Omega)$ be the set of C^∞-functions with compact support in Ω, then $W_0^{m,2}(\Omega)$ means the closure of $C_0^\infty(\Omega)$ in $W^{m,2}(\Omega)$ with respect to $\|\ \|_{m,2}$. Finally we set for $m \in \mathbb{Z}^+$, s_m to be the cardinal number of the set $S_m :=$ $:= \{\alpha \mid \alpha \in \mathbb{Z}^{+N}, |\alpha| \le m\}$, and $\xi_m(u)(x) :=$ $:= (D^\alpha u(x))_{|\alpha| \le m}$.

With these notations we can state the assumptions, we will make in this section.

(H1) $m, N \in \mathbb{N}$. $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, such that the natural embeddings of $W^{m,2}(\Omega)$ in $W^{n,2}(\Omega)$ are completely continuous for $0 \le n < m$. Further suppose that for $\alpha, \beta \in S_m$ $a_{\alpha\beta} \in L^\infty(\Omega)$ and $a_{\alpha\beta} = a_{\beta\alpha}$.

(H2) V is a closed, linear subspace of $W^{m,2}(\Omega)$, which contains $W_0^{m,2}(\Omega)$. $a: V \times V \longrightarrow \mathbb{R}$, defined by

$$a(u,v) := \sum_{|\alpha|, |\beta| \le m} \int_\Omega a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) dx$$

is uniformly, strongly elliptic.

Then, assuming (H1) and (H2), a continuous selfadjoint, linear Fredholm operator $L: V \longrightarrow V$ is defined by: $\langle Lu,v\rangle_{m,2} = a(u,v)$ for $u,v \in V$.

In dealing with the resonance case, we further suppose:

(H3) $\dim(\ker(L)) > 0$.

Concerning the nonlinear part we make the following hypotheses:

(H4) For $\alpha \in S_m$ $g_\alpha : \Omega \times \mathbb{R}^{s_m} \longrightarrow \mathbb{R}$ satisfies Carathéodory's conditions, i.e. for each $y \in \mathbb{R}^{s_m}$ $g_\alpha(\cdot, y)$ is measurable in Ω, and for $x \in \Omega$ (a.e.) $g_\alpha(x,\cdot)$ is continuous. Further the following growth restriction is assumed. There is a $c > 0$, $\sigma \in [0,1)$ and $\Theta \in L^2(\Omega)$, such that

$$|g_\alpha(x,y)| \leq c \sum_{|\beta| \leq m} |pr_\beta(y)|^\sigma + \Theta(x)$$

is satisfied for each $y \in \mathbb{R}^{s_m}$ and $|\alpha| \leq m$, and for $x \in \Omega$ (a.e.).

(H5) For $\alpha \in S_m$ there is a measurable function $h_\alpha :$ $: \Omega \times \Sigma \longrightarrow \mathbb{R}$, where $\Sigma := \{y \mid y \in \mathbb{R}^{s_m}, |y| = 1\}$, and $\Theta_\alpha \in L^{2/1-\sigma}(\Omega)$ with: $|h_\alpha(x,y)| \leq \Theta_\alpha(x)$ for $x \in \Omega$ (a.e.) and all $y \in \Sigma$, and: If $(y_n)_{n\in\mathbb{N}} \in \Sigma^{\mathbb{N}}$ with $y_n \longrightarrow y$ and $(\varphi_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $\varphi_n \longrightarrow \infty$, then for all $\alpha \in S_m$ and for $x \in \Omega$ (a.e.) we have:

$$\lim_{n \to \infty} g_\alpha(x, \varphi_n y_n)/\varphi_n^\sigma = h_\alpha(x,y).$$

We set for $v: \Omega \longrightarrow \mathbb{R}$: $\Omega_0(v) := \{x \mid x \in \Omega,\ v(x) \neq 0\}$.

(H6) For all $w \in \mathrm{Ker}(L)$ with $\|w\|_{m,2} = 1$ and all $\alpha \in S_m$

$$\int_{\Omega_0(D^\alpha w)} h_\alpha(s, \xi_m(w)(x)/|\xi_m(w)(x)|)\,|\xi_m(w)(x)|^\sigma D^\alpha w(x)dx \geq 0,$$

and for at least one $\alpha \in S_m$ the integral is strictly greater than zero.

(H7) There is a $k \in \mathbb{R}^+$ with $k < \inf\{|\lambda| \mid \lambda \in \sigma_e(L)\}$, such that

$$|g_\alpha(x,z,y_1) - g_\alpha(x,z,y_2)| \leq k\,|y_1 - y_2|$$

for all $\alpha \leq S_m$, for $x \in \Omega$ (a.e.), for all $z \in \mathbb{R}^{s_{m-1}}$ and all $y_1, y_2 \in \mathbb{R}^{s_m - s_{m-1}}$.

If (H4) is satisfied, we define a generalized Dirichlet form by:

$$n(u,v) := \sum_{|\alpha| \leq m} \int_\Omega g_\alpha(x, \xi_m(u)(x))D^\alpha v(x)dx \text{ for } u,v \in V,$$

i.e. the nonlinear part is given in divergence form. It is well-known that a continuous operator $N\colon V \longrightarrow V$ is given by:

$$\langle Nu, v\rangle_{m,2} = n(u,v) \text{ for } u,v \in V.$$

In the sequel we are concerned with the following boundary value problem:

Find a solution u of

$\otimes$ $a(u,v) = n(u,v)$ for all $v \in V$.

For a discussion about the type of this problem, we refer to [1] and mention only that $V = W_0^{m,2}(\Omega)$ leads to the Dirichlet boundary conditions.

Further, in regard to [4], we notice that, when the boundary of Ω is suitable, we can also treat $\otimes$ for a linear Dirichlet form:

$$A(u,v) = a(u,v) + \sum_{|\alpha|,|\beta| \leq m-1} \int_{\partial\Omega} A_{\alpha\beta}(x)D^\alpha u(x)D^\beta v(x)dS,$$

where $A_{\alpha\beta} \in L^{\infty}(\partial\Omega)$ and dS is defined, as in [12] chap. 3.

We obtain the following existence assertion for $\otimes$:

<u>Theorem 3</u>: Let (H1) - (H7) be satisfied.

Then there exists a solution $u \in V$ of $\otimes$.

The proof will be given in three steps:

<u>Lemma 1:</u> Suppose that the assumptions on Ω and V in (H1) and (H2) are satisfied, and (H4) and (H7) are fulfilled.

Then N is a k-set-contraction.

<u>Proof.</u> As mentioned before, we know the continuity of N. For $\alpha \in S_m$ we set N_α by:

$$\langle N_\alpha u, v \rangle_{m,2} = \int_\Omega g_\alpha(x, \xi_m(u)(x)) D^\alpha v(x) dx.$$

If $|\alpha| < m$, N_α is completely continuous. Therefore we are done, if $\tilde{N} := \sum_{|\alpha| = m} N_\alpha$ is a k-set-contraction. Let $Z := L^2(\Omega, R^{s_m - s_{m-1}})$ and $\psi \in Z$. The map $T_{\alpha,\psi} : V \longrightarrow L^2(\Omega)$, defined by $T_{\alpha,\psi}(u) = g_\alpha(\cdot\, , \xi_{m-1}(u)(\cdot), \psi)$, is completely continuous with respect to $\| \ \|_{m,2}$ on V and $\| \ \|_{o,2}$ on $L^2(\Omega)$, since (H1) ensures the complete continuity of $u \longmapsto (\xi_{m-1}(u), \psi)$ from V into $(L^2(\Omega))^{s_m}$, and because the Nemyckij operator, induced by g_α , is continuous from $(L^2(\Omega))^{s_m}$ into $L^2(\Omega)$. Further the uniform equicontinuity of the family $\{u \longmapsto (\xi_{m-1}(u), \psi)\}_{\psi \in Z}$ and the complete continuity of each map $u \longmapsto (\xi_{m-1}(u), \psi)$ imply that $(T_{\alpha,\psi})_{\psi \in Z}$ is uniformly equicontinuous on bounded sets.

Now, let $B \subseteq V$ be bounded and $\varepsilon > 0$. Then there exists a finite covering $(B_1, \ldots, B_n)$ of B by subsets of B with $\operatorname{diam}_{m,2}(B_i) \le \gamma(B) + \varepsilon/2$ for $1 \le i \le n$. Further, using the above stated properties of $(T_{\alpha,\psi})_{\psi\in Z}$, we obtain for $i \in \{1,\dots,n\}$ a covering $(C_1^i,\dots,C_j^i)$ of B_i by subsets with $\operatorname{diam}_{o,2}(T_{\alpha,\psi}(C_\mu^i)) \leq \varepsilon/2s_m$ for $1 \leq \mu \leq j_i$ and $\psi \in Z$. Hence we can suppose that the covering $(B_1,\dots,B_n)$ additionally satisfies:

$$\operatorname{diam}_{o,2}(T_{\alpha,\psi}(B_i)) \leq \varepsilon/2s_m \text{ for } 1\leq i\leq n \text{ and } \psi \in Z.$$

Then we have:

$$\| \tilde{N}u - \tilde{N}v \|_{m,2} =$$

$$= \sup_{\|w\|_{m,2}=1} \Big| \sum_{|\alpha|=m} \int_\Omega [g_\alpha(x, \xi_m(u)(x)) -$$

$$- g_\alpha(x, \xi_m(v)(x))]\, D^\alpha w(x)dx \Big| =$$

$$= \sup_{\|w\|_{m,2}=1} \Big| \sum_{|\alpha|=m} \int_\Omega [g_\alpha(x, \xi_{m-1}(u)(x), \eta_m(u)(x)) -$$

$$- g_\alpha(x, \xi_{m-1}(v)(x), \eta_m(v)(x))]\, D^\alpha w(x)dx \Big| ,$$

where $\eta_m(u)(x) := (D^\alpha u(x))_{|\alpha|=m}$. Then we obtain:

$$\| \tilde{N}u - \tilde{N}v \|_{m,2} \leq \sup_{\|w\|_{m,2}=1} \sum_{|\alpha|=m} [\int_\Omega | g_\alpha(x, \xi_{m-1}(u)(x),$$

$$\eta_m(u)(x)) - g_\alpha(x, \xi_{m-1}(v)(x), \eta_m(v)(x)| \, |D^\alpha w(x)| \, dx] \leq$$

$$\leq \sup_{\|w\|_{m,2}=1} \sum_{|\alpha|=m} [\int_\Omega |g_\alpha(x), \xi_{m-1}(u)(x), \eta_m(u)(x)) -$$

$$- g_\alpha(x, \xi_{m-1}(u)(x), \eta_m(v)(x))| \, | D^\alpha w(x)| \, dx +$$

$$+ \int_\Omega | g_\alpha(x, \xi_{m-1}(u)(x), \eta_m(v)(x)) - g_\alpha(x, \xi_{m-1}(v)(x),$$

$$\eta_m(v)(x))| \, | D^\alpha w(x)| \, dx] \leq$$

$$\leq \sup_{\|w\|_{m,2}=1} \sum_{|\alpha|=m} [k \int_\Omega | \eta_m(u)(x) - \eta_m(v)(x)| .$$

$$. |D^\alpha w(x)| \, dx + \int_\Omega | g_\alpha(x, \xi_{m-1}(u)(x), \eta_m(v)(x)) -$$

$- g_\alpha(x, \xi_{m-1}(v)(x), \eta_m(v)(x))| |D^\alpha w(x)|dx]$

$\leq k \sup_{\|w\|_{m,2}=1} \sum_{|\alpha|=m} [\int_\Omega |\eta_m(u)(x) -$

$- \eta_m(v)(x)|^2 dx]^{1/2} [\int_\Omega |D^\alpha w(x)|^2 dx]^{1/2} +$

$+ \sup_{\|w\|_{m,2}=1} \sum_{|\alpha|=m} [\int_\Omega |g_\alpha(x, \xi_{m-1}(u)(x), \eta_m(v)(x)) -$

$- g_\alpha(x, \xi_{m-1}(v)(x), \eta_m(v)(x))|^2 dx]^{1/2} [\int_\Omega |D^\alpha w(x)|^2 dx]^{1/2} \leq$

$\leq k\|u - v\|_{m,2} + (s_m - s_{m-1})\varepsilon/2s_m \leq k\gamma(B) + \varepsilon/2 + \varepsilon/2.$

Hence $\mathrm{diam}_{m,2}(\tilde{N}(B_i)) \leq k\gamma(B) + \varepsilon$ for $1 \leq i \leq n$ and $\varepsilon > 0$, therefore, $\gamma(\tilde{N}(B)) \leq k\gamma(B)$.

<u>Lemma 2:</u> Let (H1) - (H6) be satisfied. Then the following assertion holds: For each bounded subset W of $R(L)$, there exists a $t_o > 0$ with: $\langle N(tw + t^\sigma z), w\rangle > 0$, for all $t \geq t_o$, all $z \in W$, and all $w \in \mathrm{Ker}(L)$ with $\|w\|_{m,2} = 1$.

<u>Proof</u>. Otherwise there exists a $W \subseteq R(L)$, $(t_n)_{n\in\mathbb{N}} \in (\mathbb{R}^+ \setminus \{0\})^{\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in \mathrm{Ker}(L)^{\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}} \in W^{\mathbb{N}}$, satisfying W is bounded, $t_n \longrightarrow \infty$, $\|w_n\|_{m,2} = 1$ for $n \in \mathbb{N}$ and:

$$\sum_{|\alpha|\leq m} \int_\Omega g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) D^\alpha w_n(x)dx \leq 0.$$

By going if necessary to subsequences, we can assume: There is $w \in \mathrm{Ker}(L)$ with: $\|w_n - w\|_{m,2} \longrightarrow 0$, $\|w_n + t_n^{\sigma-1}v_n - w\|_{m,2} \longrightarrow 0$, $D^\alpha w_n(x) \longrightarrow D^\alpha w(x)$ for $x \in \Omega$ (a.e) and each $\alpha \in S_m$, and $D^\alpha w_n(x) + t_n^{\sigma-1}D^\alpha v_n(x) - D^\alpha w(a) \longrightarrow 0$ for $x \in \Omega$ (a.e.) and $\alpha \in S_m$. Now let $\alpha \in S_m$.

$$t_n^{-\sigma} \int_\Omega g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) D^\alpha w_n(x)dx =$$

$$= t_n^{-\sigma} \int_\Omega g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) D^\alpha w(x)dx +$$

$$+ t_m^{-\sigma} \int_\Omega g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) [D^\alpha w_n(x) -$$

$$- D^\alpha w(x)]\, dx =: I_n + II_n$$

We claim: $\lim_{n\to\infty} II_n = 0.$

$$|II_n| \le [\int_\Omega |t_n^{-\sigma} g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x))|^2 dx]^{1/2} \cdot$$

$$\cdot \| D^\alpha w_n - D^\alpha w \|_{0,2}.$$

Since $\| D^\alpha w_n - D^\alpha w \|_{0,2} \longrightarrow 0$, we are done, if the integral is bounded. Using the growth condition in (H4), we find:

$$\int_\Omega |t_n^{-\sigma} g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x))|^2 dx \le$$

$$= \int_\Omega [c \sum_{|\beta|=m} | D^\beta (w_n + t_n^{\sigma-1} v_n)(x))| + t_n^{-\sigma} \Theta(x)]^2 dx \le$$

$$\le 2c^2 s_m \sum_{|\beta|=m} \int_\Omega | D^\beta (w_n + t_n^{\sigma-1} v_n)(x)|^{2\sigma} dx +$$

$$+ 2t_n^{-2\sigma} \| \Theta \|_{0,2}^2 \le 4c^2 s_m \| w_n + t_n^{\sigma-1} v_n \|_{m,2}^2 + 2t_n^{-2\sigma} \| \Theta \|_{0,2}^2 +$$

$$+ 2s_m \operatorname{meas}(\Omega).$$

Since $(w_n + t_n^{\sigma-1} v_n)_{n\in\mathbb{N}}$ is a $\| \ \|_{m,2}$ convergent sequence, the boundedness is proved.

We have for I_n:

$$I_n = t_n^{-\sigma} \int_{\Omega_0(D^\alpha w)} g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) D^\alpha w(x)dx.$$

Since $D^\alpha w_n(x) + t_n^{\sigma-1} D^\alpha v_n(x) \longrightarrow D^\alpha w(x)$ a.e. in Ω, we find for almost each $x \in \Omega_0(D^\alpha w)$ an $n_0(x) \in \mathbb{N}$ with:

$$D^\alpha w_n(x) + t_n^{\sigma-1} D^\alpha v_n(x) \neq 0 \text{ for } n \ge n_0(x).$$

Hence $|\xi_m(w_n + t_n^{\sigma-1} v_n)(x)| > 0$ for almost every $x \in \Omega_o(D^\alpha w)$ and all $n \geq n_o(x)$. Thus $\lim_{n\to\infty} |\xi_m(t_n w_n + t_n^\sigma v_n)(x)| = \infty$ holds in $\Omega_o(D^\alpha(w))$ a.e. . Therefore (H5) implies for almost each $x \in \Omega_o(D^\alpha w)$:

$$\oplus \lim_{n\to\infty} t_n^{-\sigma} g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) =$$

$$= h_\alpha(x, \xi_m(w)(x)/|\xi_m(w)(x)|) . |\xi_m(w)(x)|^\sigma .$$

(For $n \geq n_o(x)$ choose $y_n = \xi_m(t_n w_n + t_n^\sigma v_n)(x)/|\xi_m(t_n w_n + t_n v_n)(x)|$ and $\rho_n = |\xi_m(t_n w_n + t_n^\sigma v_n)(x)|$.) Now $\oplus$ and the boundedness of $(t_n^{-\sigma} g_\alpha(\cdot, \xi_m(t_n w_n + t_n^\sigma v_n)))_{n\in\mathbb{N}}$ in $L^2(\Omega)$ involve the weak convergence of $(t_n^{-\sigma} g_\alpha(\cdot, \xi_m(t_n w_n + t_n^\sigma v_n)))_{n\in\mathbb{N}}$ to $h_\alpha(\cdot, \xi_m(w)/|\xi_m(w)|) . |\xi_m(w)|^\sigma$ in $L^2(\Omega_o(D^\alpha w))$. Hence

$$\lim_{n\to\infty} I_n = \int_\Omega h_\alpha(x, \xi_m(w)(x)/|\xi_m(w)(x)|)\, |\xi_m(w)(x)|^\sigma . D^\alpha w(x)dx.$$

Then (H6) implies:

$$\lim_{n\to\infty} \sum_{|\alpha|\leq m} \int_\Omega t_n^{-\sigma} g_\alpha(x, \xi_m(t_n w_n + t_n^\sigma v_n)(x)) D^\alpha w_n(x)dx > 0,$$

which is a contradiction.

Now we can prove Theorem 3:

Proof to Theorem 3: We realize the hypotheses of Theorem 1 for $X = Y = (V, \|\ \|_{m,2})$ and for the above defined L and N. Since L is a selfadjoint Fredholm operator, R(L) is equal to $\text{Ker}(L)^\perp$. Hence we can take the orthogonal projector on Ker(L) to be P and Q, and J to be the identity of Ker(L). The condition "N is a k-set-contraction with $k < l(L)$"

follows by Lemma 1, Theorem 2 and assumption (H7).

We derive the hypotheses (1) - (3).

(1): Let $u \in V$ and $\sigma > 0$.

$$\| Nu \|_{m,2} = \sup_{\|v\|_{m,2}=1} \left| \int_\Omega \sum_{|\alpha| \le m} g_\alpha (x, \xi_m(u)(x)) D^\alpha v(x) dx \right|$$

$$\le \sup_{\|v\|_{m,2}=1} \sum_{|\alpha|\le m} [\int_\Omega | g_\alpha (x, \xi_m(u)(x))|^2 dx]^{1/2} \cdot \| D^\alpha v \|_{0,2}$$

$$\le \sup_{\|v\|_{m,2}=1} [\int_\Omega \sum_{|\beta| \le m} (c \,| D^\beta u(x) |^\sigma + \Theta (x))^2 dx]^{1/2} \cdot \sum_{|\alpha|\le m} \| D^\alpha v \|_{0,2}$$

$$\le [2c^2 \int_\Omega (\sum_{|\beta|\le m} | D^\beta u(x)|^{2\sigma} dx + 2 \|\Theta\|^2_{0,2}]^{1/2}$$

$$\le [2c^2 s_m \sum_{|\beta|\le m} \int_\Omega | D^\beta u(x)|^{2\sigma} dx]^{1/2} + \sqrt{2}\, \|\Theta\|_{0,2}$$

$$\le \sqrt{2}c \sqrt{s_m}\, \tilde{c}\, \| u \|^\sigma_{m,2} + \sqrt{2}\, \|\Theta\|_{0,2} \le \tilde{\mu} \| u \|^\sigma_{m,2} + \tilde{\vartheta},$$

where $\tilde{\mu}$, $\tilde{\vartheta}$ are suitable constants, and $\tilde{c}$ satisfies: $|\!|\!| \varphi |\!|\!| \le c \, \|\varphi\|_{0,2}$ for each $\varphi \in L^2(\Omega)$. Here $|\!|\!| \;\; |\!|\!|$ means the quasinorm, given by $|\!|\!| \varphi |\!|\!| := (\int_\Omega | \varphi (x)|^{2\sigma} dx)^{1/2\sigma}$, which is weaker on $L^2(\Omega)$ than $\| \;\; \|_{0,2}$. Now the boundedness of the pseudo-inverse K_p ensures $\| \hat{N} u \|_{m,2} \le \mu \| u \|^\sigma_{m,2} + \vartheta$ for suitable μ, ϑ and $\sigma \in [0,1)$ (the case $\sigma = 0$ is obvious), where $\hat{N} = K_p \circ (I - Q) \in N$.

(2): Let W be a bounded subset of $R(L)$, then, using Lemma 2, there exists a $t_0 \ge 0$ with:

(∗) $\langle N(tw + t^\sigma v, w \rangle > 0$ for $t \ge t_0$, $w \in \mathrm{Ker}(L)$, $\| w \|_{m,2} = 1$ and $v \in W$.

This implies $Q \circ N(tw + t^\sigma v) \ne 0$ for $t \ge t_0$, $w \in \mathrm{Ker}(L)$,

$\| w \|_{m,2} = 1$ and $v \in W$.

(3): Set $v = 0$ in $(*)$, then $\langle Q \circ N(tw), tw \rangle =$ $= t \langle N(tw), w \rangle > 0$ for $t \geq t_o$. Therefore the Poincaré-Bohl theorem implies:

$$\deg (J \circ Q \circ N \mid Ker(L), \{w \mid w \in Ker(L), \| w \|_{m,2} < t\}, 0) \neq 0.$$

Now Theorem 1 yields: There is $u \in V$ with: $Lu - Nu = 0$, which implies: $a(u,v) = n(u,v)$ for all $v \in V$.

4. Here we will make a few notes on Theorem 3: Concerning the linear part we only mention:

Remark 1: The spectrum of L must be determined with respect to V, i.e. we have to consider an equation of the form:

$$\sum_{\alpha,\beta \in S_m} \int_\Omega a_{\alpha\beta}(x) D^\alpha u(x) D^\beta v(x) dx = \lambda \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u(x) D^\alpha v(x) dx$$

for $v \in V$. In regard to (H1) and (H2) we can suppose without loss of generality: $a_{\alpha\beta} = 0$ for $|\alpha| + |\beta| < 2m$.

If we consider the Laplacian for example, i.e.

$$\langle \Delta u, v \rangle_{m,2} = \sum_{1 \leq i \leq N} \int_\Omega D^i u(x) D^i v(x) dx,$$

we obtain $\sigma_e(\Delta) = \{1\}$, hence $\ell(\Delta) = 1$.

Assumptions, analogous to (H4) - (H6), appear in [3].

Remark 2: One observes that the choice of σ is unique, because it depends not only on the growth condition in (H4), but also on (H5) - (H6). Instead of $\otimes$ we can consider:

$(\times\times)$ $\quad a(u,v) = n(u,v) + \langle f,v \rangle_{0,2}$ for $v \in V$,

where f is given function of $L^2(\Omega)$, by setting $\tilde{g}_0 = g_0 + f$. If $\sigma > 0$, we obtain:

Corollary: Let (H1) - (H7) be satisfied and $\sigma > 0$. For $f \in L^2(\Omega)$ there exists a solution $u \in V$ of $(\times\times)$.

We end with some special cases:

Remark 3: If g_α depends only on x and the α-th derivative (we write then $g_\alpha(x, D^\alpha u(x))$), the conditions (H5) and (H6) are reduced to:

(H5)' There exists functions $h_\alpha^+ \in L^{2/1-\sigma}$ and $h_\alpha^- \in L^{2/1-\sigma}$ with:

$$\lim_{y \to \pm\infty} g_\alpha(x,y)/|y|^\sigma = h_\alpha^\pm(x) \text{ for } x \in \Omega \text{ (a.e.)}.$$

(H6)' For all $w \in \mathrm{Ker}(L)$ with $\|w\|_{m,2} = 1$ and all $\alpha \in$ $\in S_m$

$$\int_{\Omega_+(D^\alpha w)} h_\alpha^+(x)\,|D^\alpha w(x)|^{1+\sigma}\,dx - \int_{\Omega_-(D^\alpha w)} h_\alpha^-(x)\,|D^\alpha w(x)|^{1+\sigma} \geq 0,$$

where $\Omega_{\binom{+}{-}}(D^\alpha w) := \{x \mid x \in \Omega\ ,\ D^\alpha w(x) \binom{>}{<} 0\}$, and at least for one $\alpha \in S_m$ the integral is strictly greater than zero.

Remark 4: If $V = W_0^{m,2}(\Omega)$ and $\sigma = 0$, Theorem 3 is a generalization of the Landesmann Lazer result (e.g. [10], [17]). When we consider equation $(\times\times)$ (see Remark 2), we receive the following condition for $|\alpha| = 0$ in (H6):

$$\int_\Omega f(x)w(x)dx \leq \int_{\Omega_0(w)} h_0(x, \xi_m(u)(x)/|\xi_m(u)(x)|)w(x)dx.$$

Remark 5: To prove Theorem 3.2 in [4] for the quasilinear case, we need the assertion of Corollary VI.6 in [11] for set-contractions, which can be derived in the same manner from Theorem 1 as the just mentioned Corollary from Theorem VI.4 there. We omit details.

Remark 6: Instead of Theorem 1 we can use a theorem for set-contractions, corresponding to Theorem 1 in [2], to prove Theorem 3.

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