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THE SPECTRAL RADII OF AN OPERATOR AND ITS MODULUS

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Abstract: The author proves an inequality connecting the spectral radius of a linear operator in a Hilbert space of finite dimension and the spectral characteristics of its modulus, the positive definite factor of its polar decomposition.

Key Words: Linear operator, Hilbert space, spectral radius, polar decomposition.

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It is the purpose of the present remark to investigate the connection between spectral properties of a linear operator and its modulus, the positive definite factor in the polar decomposition.

The basic result is an inequality connecting the spectral radii of a positive definite operator P and of the operator UP where U is an arbitrary unitary operator (Lemma (2,1) of the present remark). As an easy consequence we obtain the main result (Theorem (3,1)).

For each positive definite M and each unitary U

$$(|M^{-1}|_{\sigma}^{-1})^{\frac{n-1}{n}} |M|_{\sigma}^{\frac{1}{n}} \leq |UM|_{\sigma} \leq |M|_{\sigma}.$$

This result has some interesting corollaries.

1. Definition and notation. If $a_1,\dots,a_m$ are positive numbers, we denote by $G(a_1,\dots,a_m)$ their geometric mean

$$G(a_1,\dots,a_m) = (a_1 a_2 \dots a_m)^{1/m}$$

In the whole paper H will be a Hilbert space of dimension n. If A is a linear operator on H we denote by $\sigma(A)$ its spectrum, by $|A|_\sigma$ its spectral radius and by $|A|$ its norm (as an operator on H) hence $|A| = (|A^*A|_\sigma)^{1/2}$.

Now suppose that A is an invertible operator on H. Then A^*A and AA^* are both positive definite; denote by $(A^*A)^{1/2}$ and $(AA^*)^{1/2}$ respectively their positive definite square roots. There exist two unitary operators U and V such that

$$A = U(A^*A)^{1/2} = (AA^*)^{1/2}V$$

and both these decompositions are unique. Hence $(A^*A)^{1/2}$ could be called the left modulus of A and $(AA^*)^{1/2}$ the right modulus of A. Speaking about the modulus of an operator we should specify which of the two possible definitions we have in mind. The operator A being invertible, the operators A^*A and AA^* have the same spectrum since $AA^* = A(A^*A)A^{-1}$. It follows that there is no ambiguity if we are dealing with spectral properties of the two moduli. In particular, the following two definitions are meaningful. We shall denote by max M(A) the maximal eigenvalue of $(A^*A)^{1/2}$ or, equivalently, of $(AA^*)^{1/2}$. It follows that $\max M(A) = |(A^*A)^{1/2}|_\sigma = |(AA^*)^{1/2}|_\sigma$. We shall denote by min M(A) the minimal eigenvalue of $(A^*A)^{1/2}$ or of $(AA^*)^{1/2}$. It follows that $\min M(A) = |(A^*A)^{-1/2}|_\sigma^{-1} = |(AA^*)^{-1/2}|_\sigma^{-1}$.

2. <u>Preliminaries</u>. The results of the present paper are based on the following fundamental proposition.

(2,1) Let D be an n-dimensional diagonal matrix with positive diagonal entries $d_1, d_2, \dots, d_n$. Denote by $\mathcal{U}$ the set of all unitary matrices of order n. Then

$$\min \{ |UD|_6 \; ; \; U \in \mathcal{U} \} = G(d_1, \dots, d_n)$$

Let T be positive definite. Then $\min \{ |UT|_6 \; ; \; U \in \mathcal{U} \}$ equals the geometric mean of the eigenvalues of T.

Proof. If T is positive definite, there exists a unitary V and a diagonal matrix D such that $T = VDV^*$. Since $|UT|_6 = |UVDV^*|_6 = |V^*UVD|_6$ it suffices to prove the first assertion.

Denote by m the minimum on the left hand side. Clearly, for each $U \in \mathcal{U}$ we have

$$|UD|_6 \geq |\det UD|^{1/n} = |\det U \det D|^{1/n} = |\det D|^{1/n} =$$
$$= G(d_1, \dots, d_n).$$

It follows that $m \geq G(d_1, \dots, d_n)$.

On the other hand, consider the matrix V defined by the relations

$v_{i,i+1} = 1$ for $i = 1, 2, \dots, n-1$

$v_{n,1} = 1$

$v_{pq} = 0$ for all remaining pairs of indices p, q

Since V is a permutation matrix, we have $V \in \mathcal{U}$. It is not difficult to show that $(VD)^n$ is a diagonal matrix, in fact that

$$(VD)^n = hI$$

where $h = d_1 d_2 \dots d_n = G(d_1,\dots,d_n)^n$. Since $|VD|_\sigma \leq$ $\leq |(VD)^n|^{1/n}$, we have

$$m \leq |VD|_\sigma \leq |(VD)^n|^{1/n} = h^{1/n} = G(d_1,\dots,d_n).$$

Together with the preceding inequality this established the lemma.

The following simple lemma is valid even in infinite dimensional Hilbert spaces.

(2,2) Let H be a Hilbert space, T a bounded linear operator on H. Then

1° for arbitrary unitary operators U and V

$$|UTV| = |T|$$

2° let M be the left or right modulus of T; then

$$|T|_\sigma \leq |M|_\sigma$$

Proof. The first assertion is obvious. To prove the second assertion, we recall that

$$|(T^*T)^{1/2}|_\sigma = |(TT^*)^{1/2}|_\sigma$$

so that we may restrict ourselves to the case of the left modulus. There exists a partial isometry U such that $T = U(T^*T)^{1/2}$ and $U^*T = (T^*T)^{1/2}$. Hence

$$|T|_\sigma = |U(T^*T)^{1/2}|_\sigma \leq |U(T^*T)^{1/2}| = |(T^*T)^{1/2}| = $$
$$= |(T^*T)^{1/2}|_\sigma$$

and the proof is complete.

(2,3) Let H be a Hilbert space of dimension n. Let A be a linear operator on H. Then

$$|A|_\sigma \geq (\max M(A))^{1/n}(\min M(A))^{n-1/n}$$

Proof. If $\min M(A) = 0$, the inequality is trivially satisfied. If min $M(A) > 0$, the operator $A^* A$ is invertible hence A is invertible. We may therefore limit ourselves to the case of an invertible A. Let B be the matrix of A in an orthonormal basis of H. Since B is invertible there exists a unitary matrix U such that $B = U(B^* B)^{1/2}$; since $(B^* B)^{1/2}$ is positive definite, there exists a unitary V such that $(B^* B)^{1/2} = VDV^*$ where D is a diagonal matrix of the form

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

Clearly we may assume that $d_1 \geq d_2 \geq \ldots \geq d_n > 0$. We have then $|A|_6 = |B|_6 = |UVDV^*|_6 = |V^* UVD|_6 \geq G(d_1, \ldots, d_n) \geq$
$\geq G(d_1, d_n, \ldots, d_n) = d_1^{1/n} d_n^{n-1/n}$. Since $d_1 = \max M(A)$ and $d_n = \min M(A)$, this completes the proof.

3. The main result.

(3,1) Theorem. Let H be a Hilbert space of dimension n. Let M be a positive definite operator on H. Then, for each unitary U on H, the following inequalities hold.

$$(|M^{-1}|_6^{-1})^{n-1/n} (|M|_6)^{1/n} \leq |UM|_6 \leq |M|_6$$

Proof. First of all,

$$|UM|_6 \leq |UM| = |M| = |M|_6 \quad .$$

The second inequality is a consequence of (2,3) and the fact that the minimal eigenvalue of M equals $|M^{-1}|_6^{-1}$.

(3,2) Corollary. Let H be a Hilbert space of dimension n. If M is a positive definite operator on H and U and V are unitary then the following inequality holds.

$$(\,|(UM)^{-1}|^{-1})^{n-1/n}(\,|UM|)^{1/n} \leq |M|_{\sigma} = |VM|$$

Proof. This time, we use the following equalities

$|M|_{\sigma} = |M| = |VM|$

$|M|_{\sigma} = |UM|$

$|M^{-1}|_{\sigma}^{-1} = |M^{-1}|^{-1} = |(UM)^{-1}|^{-1}$

(3,3) <u>Corollary</u>. Let A be a linear operator on the n-dimensional Hilbert space H. If A is invertible then

$$|A|^{1/n}(|A^{-1}|^{-1})^{n-1/n} \leq |A|_{\sigma} \leq |A|$$

Proof. This is an immediate consequence of the main theorem and of the following equalities.

$|M|_{\sigma} = |M| = |UM| = |A|$

$|M^{-1}|_{\sigma} = |M^{-1}| = |M^{-1}U^{*}| = |A^{-1}|$

As an immediate consequence, we have the following inequality obtained recently by N.J. Young in the course of his investigations of the critical exponent of n-dimensional Hilbert space. The result of Young represents a considerable improvement of an inequality proved previously by Daniel and Palmer.

(3,4) Let A be an invertible linear operator on an n-dimensional Hilbert space. Then

$$|A| \leq |A|_{\sigma}^{n}|A^{-1}|^{n-1}$$

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