Ladislav Bican

Splitting of pure subgroups

Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 2, 241--249

Persistent URL: http://dml.cz/dmlcz/105690

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,2 (1976)

SPLITTING OF PURE SUBGROUPS

Ladislav BICAN, Praha

Abstract: This note gives a structural characterization of torsion-free abelian groups H of finite rank n having the property: if G is a mixed group with $G/T \cong H$ then every pure subgroup of G of rank n splits if and only if G satisfies Conditions (α), (γ).

Key Words: Splitting group, p-rank, regular subgroup, generalized regular subgroup.

AMS: Primary 20K15 Ref. Ž.: 2.722.1
Secondary 20K25, 20K99

By the word "group" we shall always mean an additively written abelian group. The symbol π will denote the set of all primes. If T is a torsion group, then T_p will denote the p-primary component of T and similarly if $\pi' \subseteq \pi$ then $T_{\pi'}$ is defined by $T_{\pi'} = \sum_{p \in \pi'} T_p$. If G is a mixed group, M a subset of G, $\pi' \subseteq \pi$ and $T_{\pi'} = 0$ then $\{M\}^G_{\pi'} = \{g \in G \mid mg \in \{M\}$ for some non-zero integer m divisible by the primes from π' only $\}$ is the π'-pure closure of M in G.

In the sequel, we shall deal with mixed groups G with the torsion part $T = T(G)$, $\overline{G}$ will denote the factor-group G/T and $\overline{a} = a + T$ for all $a \in G$. If H is a torsionfree group then the set of all elements g of H having infinite p-height is a subgroup of H which will be denoted by $H[p^{\infty}]$. Any maximal linearly independent set of elements of a torsionfree group H is called basis. It is well-known (see [7]) that if H is a torsionfree group and K its free subgroup of the same rank then the number $r_p(H)$ of summands $C(p^{\infty})$ in H/K does not depend on the particular choice of K and this number is called the p-rank of H. A subgroup K of a torsionfree group H is called regular if every element of K has in K the same type as in H and it is called generalized regular if for every $g \in K$ the characteristic of g in K and in H differ only in finitely many places. Other notation and terminology is essentially that of [4] and we shall freely use the results of [1] and [3].

Now we shall formulate Conditions (α), (γ) (see [1]).
Condition (α): A mixed group G with the torsion part T satisfies Condition (α) if to any $g \in G \dot{-} T$ there exists an integer m such that mg has in G the same type as $\bar{g}$ in $\bar{G}$.
Condition (γ): We say that a mixed group G with the torsion part T satisfies Condition (γ) if it holds: If $\bar{G} = G/T$ contains a non-zero element of infinite p-height, then T_p is a direct sum of a divisible and a bounded group.

<u>Lemma 1</u>: Let G be a mixed group of the form $G = \sum_{i=1}^{\infty} \{b_i\} \oplus H$, where $\{b_i\}$ is a cyclic group of order p^{l_i}, $l_i < l_{i+1}$, $i = 1,2,\dots$ and H is a torsionfree group of rank n such that $H[p^{\infty}] \neq 0$. Then G contains a non-splitting pure subgroup of rank n.

Proof: Let $\{a, h_2, \dots, h_n\}$ be a basis of H such that a is of infinite p-height. Put $K = \{a, h_2, \dots, h_n\}^{G}_{\pi \dot{-} \{p\}} +$

$+ \{h_2, \ldots, h_n\}_p^G$ and let $a_i \in H$ be such elements that $p^{l_i} a_i = a$. Obviously, $H = \{K, a_1, a_2, \ldots\}$. Put $s_i = a_i + b_i$, $i = 1, 2, \ldots$, $U = \{K, s_1, s_2, \ldots\}$ and $S = \{U\}^G_{\pi - \{p\}}$.

First, we shall prove the purity of S in G. It suffices to show that any equation $p^k x = u$, $u \in U$, solvable in G is solvable in U, since the equation $p^k x = s$, $s \in S$ is solvable in G then $p^k mx = ms$, $ms \in U$ for a suitable non-zero integer m prime to p. Hence $p^k u' = ms$ for some $u' \in U$ and the equality $p^k \varrho + m\sigma = 1$ yields $s = p^k(\varrho s + \sigma u')$, $\varrho s + \sigma u' \in S$. So, let the equation $p^k x = u$, $u \in U$, be solvable in G, $x = \sum_{i=1}^{l} \mu_i b_i + h$. Then $p^k x = p^k(\sum_{i=1}^{l} \mu_i b_i + h) = u = h' + \sum_{i=1}^{l} \lambda_i s_i$, $h' \in K$, hence $p^{l_i} \mid (\lambda_i - p^k \mu_i)$ and $p^k h = h' + \sum_{i=1}^{l} \lambda_i a_i$. Thus there are integers ν_i, $i = 1, 2, \ldots, l$, with $\lambda_i = p^k \mu_i + p^{l_i} \nu_i$. Put $\nu = \sum_{i=1}^{l} \nu_i$. Since $h' \in K$, $h' = h_1 + h_2$, where $m h_1 = \varrho a + \sum_{i=2}^{n} \varrho_i h_i$ for some m prime to p and $p^r h_2 = \sum_{i=2}^{n} \sigma_i h_i$. Hence $mp^{k+r} h = p^r \varrho a + p^r \sum_{i=2}^{n} \varrho_i h_i + m \sum_{i=2}^{n} \sigma_i h_i + p^r m \sum_{i=1}^{l} \lambda_i a_i$. Since $p^r \varrho a + p^r m \sum_{i=1}^{l} \lambda_i a_i$ is of infinite p-height, $p^{k+r} v = p^r \sum_{i=2}^{n} \varrho_i h_i + m \sum_{i=2}^{n} \sigma_i h_i$, $v \in K$. Put $u' = m \sum_{i=1}^{l} \mu_i s_i + p^{l_j - k}(m\nu + \varrho) s_j + v \in U$, $l_j \geq k$. Now for $p^k \alpha + m\beta = 1$ we have $h' + \sum_{i=1}^{l} \lambda_i s_i = p^k \alpha (h' + \sum_{i=1}^{l} \lambda_i s_i) + \beta m (h' + \sum_{i=1}^{l} \lambda_i s_i) = p^k(\alpha(h' + \sum_{i=1}^{l} \lambda_i s_i) + \beta u') \in p^k U$ since $p^k u' = m \cdot \sum_{i=1}^{l} \lambda_i s_i + \varrho a + \sum_{i=2}^{n} \varrho_i h_i + mh_2 = m(\sum_{i=1}^{l} \lambda_i s_i + h')$. The purity of S in G is proved.

Suppose now that S splits, $S = P \oplus B$. Then $a = t + b$, $t \in P$, $b \in B$, since $a = p^{l_1} s_1 \in S$. a is of infinite p-height in G, hence in S and hence t is of infinite p-height. However, $P \subseteq \sum_{i=1}^{\infty} \{b_i\}$ yields $t = 0$ and $a \in B$. The purity of B in G guarantees the existence of $c_j \in B$ with $p^{\ell_j} c_j = a$. All c_j, $j = 1,2,\ldots$ are of infinite p-height, hence $c_j - a_j \in \sum_{i=1}^{\infty} \{b_j\}$ are of infinite p-height and consequently $c_j = a_j$, $j = 1,2,\ldots$. In particular, we have $a_1 = c_1 \in B \subseteq S$ and hence $b_1 = s_1 - a_1 \in S$.

By the definition of S, $mb_1 \in U$ for some integer $m \neq 0$ prime to p. Thus $mb_1 = v + \sum_{i=1}^{\ell} \lambda_i s_i$, $v = v_1 + v_2 \in K$, where $m' v_1 = \rho a + \sum_{i=2}^{n} \rho_i h_i$ for some m' prime to p and $p^r v_2 = \sum_{i=2}^{n} \sigma_i h_i$. From the equality $mb_1 = v + \sum_{i=1}^{\ell} \lambda_i a_i + \sum_{i=1}^{\ell} \lambda_i b_i$, we get $p^{\ell_1} \mid (m - \lambda_1)$ and consequently $(p, \lambda_1) = 1$. Moreover, $\lambda_i = p^{\ell_i} \lambda'_i$ $i = 2,\ldots,l$. Putting $\lambda = \sum_{i=2}^{\ell} \lambda_i$ and multiplying by $p^{\ell_1+\varkappa} m'$ we obtain $0 = p^{\ell_1+\varkappa} \rho a + p^{\ell_1+\varkappa} \sum_{i=2}^{n} \rho_i h_i + p^{\ell_1} m' \sum_{i=2}^{n} \sigma_i h'_i + \lambda_1 p^r m' a + \lambda p^{\ell_1+\varkappa} m' a$. Since $\{a, h_2, \ldots, h_n\}$ is a basis, $p^{\ell_1+\varkappa} \rho + \lambda_1 p^r m' + \lambda p^{\ell_1+\varkappa} m' = 0$, hence $p \mid \lambda_1$ - a contradiction showing that S does not split.

<u>Lemma 2</u>: Let H be a torsionfree group of finite rank n satisfying the following two conditions:

(a) $r_p(H) = r(h[p^{\infty}])$ for almost all primes and for all primes p with $r(H[p^{\infty}]) = 0$,

(b) for every generalized regular subgroup K of H of rank $k \leq n$, the torsion part of the factor-group H/K has only a finite number of non-zero primary components.

If a mixed group G with $\overline{G} \cong H$ satisfies Conditions (α),(γ) then every pure subgroup of G of rank k splits.

Proof: Let S be a pure subgroup of G of rank k and $P = T \cap S$ be its torsion part. By [1, Lemma 6], S satisfies

Condition (α) and $\overline{S}$ is isomorphic to some regular subgroup of $\overline{G}$. Moreover, by [1, Lemma 10], S satisfies Condition (γ). If U is a pure subgroup of H then by [7, Theorem 6] $r_p(H) = r_p(U) + r_p(H/U)$, which together with the obvious inequality $r(H[p^\infty]) \leq r(U[p^\infty]) + r(H/U[p^\infty])$ yields $r_p(U) = r(U[p^\infty])$ for all those primes p for which $r_p(H) = r(H[p^\infty])$. It follows now easily that $r_p(\overline{S}) = r(\overline{S}[p^\infty])$ for almost all primes and for all primes p with $r(S[p^\infty]) = r(H[p^\infty]) = 0$. So the set $\pi' = \{p \in \pi \; ; \; r_p(\overline{S}) = r_p(\overline{S}[p^\infty])\}$ is cofinite and $P_{\pi \dot{-} \pi'}$ is a direct sum of a divisible and a bounded group by the hypothesis. Hence $S = P_{\pi \dot{-} \pi'} \oplus S'$. Now $S' \otimes R_{\pi'}$ splits, $S' \otimes R_{\pi'} = P' \oplus S''$, since it clearly satisfies Condition (i) of [3, Theorem]. Moreover, S' is $R_{\pi'}$-flat so that the map $S' \cong S' \otimes Z \hookrightarrow S' \otimes R = P' + S''$ is monic. Since $P' \subseteq S'$, S' splits as desired.

Lemma 3: Let H be a torsionfree group of finite rank n. If $0 \neq r(H[p^\infty]) < r_p(H)$ for every p from an infinite set π' of primes then H contains a regular subgroup K with $H[p^\infty] \subseteq K$ for all $p \in \pi'$ and $H/K = \dot{\sum}_{p \in \pi'} C(p^\infty)$.

Proof: Obviously, there is a subgroup L of H such that $H[p^\infty] \subseteq L$ for all $p \in \pi'$ and $H/L = \dot{\sum}_{p \in \pi'} C[p^\infty]$. If we order all the primes from π' in a sequence $p_1, p_2, \ldots$ and all the elements from $H \dot{-} L$ in a sequence $a_1, a_2, \ldots$, then it is easy to see that for every natural integer m there is a subgroup K_m with $\{L, \{a_1\}_\pi^H, \ldots, \{a_m\}_\pi^H\} \subseteq K_m$ and $H/K_m = C(p_m^\infty)$. If we put $K = \bigcap_{m=1}^\infty K_m$ then it is an easy exercise to show that K has all the desired properties.

Lemma 4: Let H be a torsionfree group of finite rank n containing a regular subgroup K with $0 \neq H[p^{\infty}] \subseteq K$ for every prime from an infinite set π' of primes, and $H/K = \sum_{p \in \pi'} C(p^{\infty})$. Then there is a mixed group G satisfying Conditions (α), (γ) such that $\overline{G} \cong H$ and G does not split.

Proof: Let $h_1, h_2, \dots, h_n$ be a basis of K. If we order all the primes from π' in a sequence $p_1, p_2, \dots$ then for every i, $j = 1,2,\dots$ there are elements $x_j^{(i)} \in H$ such that $p_i^j x_j^{(i)} = \sum_{r=1}^{n} \lambda_{ir}^{(j)} h_r$ where $(\lambda_{ir}^{(j)})$ $r = 1,2,\dots,n$ are p_i-adic integers. Let s_i be such that $x_1^{(i)}, \dots, x_{s_i}^{(i)} \in K$ and $x_{s_i+1}^{(i)}, \dots \notin K$. Obviously, $H = \{K, x_j^i,\ i = 1,2,\dots,\ j = s_i + 1, \dots\}$. If we denote $u_{j-s_i}^{(i)} = p_i^{j-s_i} x_j^{(i)}$, $j > s_i$ then it is easy to see that $u_j^{(i)}$ are of zero p_i-height in K for all $j = 1,2,\dots$. Further, for every $i,j = 1,2,\dots$,

$$p_i^{s_i}(u_{j+1}^{(i)} - u_j^{(i)}) = \sum_{r=1}^{n} (\lambda_{ir}^{(s_i+j+1)} - \lambda_{ir}^{(s_j+j)}) h_r = p_i^{s_i+j} v_j^{(i)} \in K$$

Define the groups

$$U = K \oplus \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \{a_j^{(i)}\},\quad X = \{v_j^{(i)} - p_i a_{j+1}^{(i)} + a_j^{(i)},\ i,j = 1,2,\dots\}$$

$$V = \{X, u_1^{(i)} - p_i a_1^{(i)},\ i = 1,2,\dots\},\quad W = \{X, p_i^{s_i+1} u_1^{(i)} - p_i^{s_i+2} a_1^{(i)},\ i = 1,2,\dots\}.$$

Then $G = U/W$ is a mixed group with the torsion part $T = V/W$ and $\overline{G} = G/T \cong U/V \cong H$, where the last isomorphism is induced by $h + \sum_{i=1}^{k} \sum_{j=1}^{k_i} \lambda_j^{(i)} a_j^{(i)} \longmapsto h + \sum_{i=1}^{k} \sum_{j=1}^{k_i} \lambda_j^{(i)} x_{j+s_i}^{(i)}$, $h \in K$ (if the last term is zero then the multiplication by $\prod_{i=1}^{k} p_i^{k_i}$ gives $p_i \mid \lambda_k^{(i)}$, $i = 1,2,\dots,k$ and the induction yields

Ker $\mathscr{S} = V$). G satisfies Conditions (α), (γ) since K is regular in H. Suppose that G splits, $G = T \oplus S$. Then S is naturally isomorphic to H and it is easily seen that $x_j^{(i)}$, $j \geq s_i$ corresponds to the element $y_j^{(i)}$ of the form $y_j^{(i)} =$ $= a_{j-s_i}^{(i)} + \sum_k \lambda_k(u_1^{(k)} - p_k a_1^{(k)}) + W$.

Further, if we denote by g_r the elements of S correponding to h_r, then $mg_r = mh_r + W$, $r = 1,2,\dots,n$, where m is a suitable non-zero integer. Now consider the equality $p_i^{s_i+1} y_{s_i+1}^{(i)} =$ $= \sum_{r=1}^{n} \lambda_{ir}^{(s_i+1)} g_r$, $(p_i,m) = 1$. Multiplying by m we get

$$p_i^{s_i+1} m(a_1^{(i)} + \sum_k \lambda_k(u_1^{(k)} - p_k a_1^{(k)}) = m \sum_{r=1}^{n} \lambda_{ir}^{(s_k+1} h_r +$$
$$+ \sum_k \mu_k(p^{s_k+1} u_1^{(k)} - p_k^{s_k+2} a_1^{(k)}).$$

If we put $\rho = \prod_k p_k^{s_k}$, $\rho_k = \rho / p_k^{s_k}$ then multiplying by ρ and comparing the coefficients we obtain

$$p_i^{s_i+1} m \sum_k \lambda_k \rho_k \lambda_{kr}^{(s_k+1)} = m \rho \lambda_{ir}^{(s_i+1)} +$$
$$+ \rho \sum_k \mu_k p_k \lambda_{kr}^{(s_k+1)}, \quad p_i^{s_i+1} m \rho \lambda_k p_k = \rho \mu_k p_k^{s_k+2} .$$

Hence $p_i \mid \mu_k p_k$ for all k and so $p_i \mid \lambda_{ir}^{(s_i+1)}$ $r =$ $= 1,2,\dots,n$, a contradiction finishing the proof.

Now we are ready to prove the main result.

Theorem 5: The following are equivalent for a torsion-free group H of finite rank n:

(i) if G is a mixed group with $\overline{G} \cong H$ then every pure subgroup of G of rank n splits if and only if G satisfies Conditions (α), (γ),

(ii) (a) $r_p(H) = r(H[p^\infty])$ for almost all primes and for all primes p with $r(H[p^\infty]) = 0$,

(b) for every generalized regular subgroup K of H of the same rank n the factor-group H/K has only a finite number of non-zero primary components.

Proof: (i) implies (ii). If $r(H[p^\infty]) = 0$, then $r_p(H) = 0$ by [3, Lemma 2 and its proof]. Condition (a) follows now from Lemmas 3, 4. As for (b), it follows easily from [3, Lemmas 3, 4].

(ii) implies (i). Let G be a mixed group with $\overline{G} \cong H$. If G satisfies Conditions (α), (γ) then every pure sungroup of G of rank n splits by Lemma 2. Conversely, if every pure subgroup of G of rank n splits then G satisfies Condition (α) by [1, Lemma 4]. If G does not satisfy Condition (γ) then for some prime p it is $r(H[p^\infty]) = 0$ and T_p is not a direct sum of a divisible and a bounded group. By the hypothesis, G splits, $G = T \oplus A$. Write $T_p = T_p' \oplus D$, D divisible, T_p' reduced. T_p' is unbounded so that it has an unbounded basic subgroup B ([1, Lemma 11]). Hence G contains a pure subgroup of the form of Lemma 1 and an application of this Lemma leads to a contradiction. Hence G satisfies Condition (γ) and the proof is complete.

R e f e r e n c e s

[1] BICAN L.: Mixed abelian groups of torsionfree rank one, Czech. Math. J. 20(95)(1970), 232-242.

[2] BICAN L.: A note on mixed abelian groups, Czech. Math. J. 21(96)(1971), 413-417.

[3] BICAN L.: Splitting in abelian groups (to appear).

[4] FUCHS L.: Abelian groups, Budapest, 1958.

[5] FUCHS L.: Infinite abelian groups I, Academic Press, 1970.

[6] MALCEV A.: Abelevy gruppy konečnogo ranga bez kručenija, Mat. Sb. 4(46)(1938), 45-68.

[7] PROCHÁZKA L.: O p-range abelevych grup bez kručenija konečnogo ranga, Czech. Math. J. 12(87)(1962), 3-43.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 20.11.1975)