Commentationes Mathematicae Universitatis Carolinae

Tomáš Kepka

Structure of triabelian quasigroups

Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 2, 229--240

Persistent URL: http://dml.cz/dmlcz/105689

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

17,2 (1976)

STRUCTURE OF TRIABELIAN QUASIGROUPS

Tomáš KEPKA, Praha

Abstract: A quasigroup is called triabelian if every its subquasigroup which is generated by at most three elements is abelian. In the present paper, some basic structural theorems on triabelian quasigroups are proved.

Key Words: Quasigroup, Moufang loop.

AMS: 20N05 Ref. Ž.: 2.722.9

As it is well known, the class of distributive quasigroups has a large number of nice algebraic properties. It is the purpose of this paper to show that the structure of triabelian quasigroups is very similar to that of distributive quasigroups. In certain sense, this paper is a continuation of the last section from [1]. First we recall some definitions. A quasigroup Q is called an RF-quasigroup (LF-quasigroup) if it satisfies the identity $bc.a. = bf(a).ca$ ($a.bc = ab.e(a)c$), where $f(a)$ and $e(a)$ is the left and the right local unit of a, resp. It is called an F-quasigroup if it is both an LF and RF-quasigroup. Further, a quasigroup Q is said to be a WA-quasigroup if $aa.bc = ab.ac$ and $bc.aa = ba.ca$ for all $a,b,c \in Q$. If moreover $ab.ca = ac.ba$ then we shall say that Q is a WAD-quasigroup. Finally, an abelian quasigroup is a quasigroup satisfying the identity $ab.cd = ac.bd$. Let Q be a quasigroup and $x \in Q$. Then L_x and R_x is the left and the right translation by x, resp. If Q is a commutative Moufang loop, then j is the identity of Q, N(Q) is the nucleous of Q and a mapping g of Q into Q is said to be nuclear if $x^{-1}.g(x) \in N(Q)$ for each $x \in Q$.

The following lemma is implicitly contained in [2].

Lemma 1. Let Q be a commutative loop and h be a mapping of Q into Q. Then the following are equivalent:

(i) $(a.h(a))(bc) = (ab)(h(a)c)$ for all $a,b,c \in Q$.

(ii) Q is a Moufang loop and h is nuclear.

Theorem 1. The following conditions are equivalent for every quasigroup Q:

(i) Q is a WA-quasigroup and there exists $a \in Q$ such that $ab.ca = ac.ba$ for all $b,c \in Q$.

(ii) Q is a WA-quasigroup and Q is isotopic to a commutative Moufang loop.

(iii) Q is a WA-quasigroup and Q is isotopic to a Moufang loop.

(iv) There are a commutative Moufang lopp $Q(\circ)$, $g,h \in \in \mathrm{Aut}\, Q(\circ)$ and $x \in Q$ such that $gh = hg$, gh^{-1} is nuclear and $ab = (g(a)\circ h(b))\circ x$ for all $a,b \in Q$.

(v) Q is a WAD-quasigroup.

Proof. (i) implies (ii). If $b,c \in Q$ then $(aa.ab)(ac.aa) = (aa.ab)(aa.ca) = (aa.aa)(ab.ca) = (aa.aa)(ac.ba) = (aa.ac)(aa.ba) = (aa.ac)(ab.aa)$. Hence $(aa.x)(y.aa) = (aa.y)(x.aa)$ for all $x,y \in Q$ and we can use [1, Proposition 4.8] and Lemma 1.

(iii) implies (iv). Let $x \in Q$ and $a \circ b = R_{xx}^{-1}(a).L_{xx}^{-1}(b)$ for all $a,b \in Q$.

As it is proved in [1], Propositions 4.1 and 4.8, $Q(\circ)$ is a CI-loop. However, $Q(\circ)$ is a Moufang loop, and hence it is commutative. The rest follows from [1 , Proposition 4.8 and Theorem 4.9].

(iv) implies (v). Since gh^{-1} is a nuclear mapping and $gh = hg$, $g^2h^{-2} = gh^{-1}gh^{-1}$ is nuclear. According to Lemma 1,

$ab.ca = (((g^2(a) \circ gh(b)) \circ g(x)) \circ ((hg(c) \circ h^2(a)) \circ h(x))) \circ x =$

$= (((g^2(a) \circ gh(b))) \circ (gh(c) \circ h^2(a))) \circ (g(x) \circ h(x))) \circ x =$

$= (((g^2(a) \circ gh(c))) \circ g(x)) \circ ((hg(b) \circ h^2(a)) \circ h(x))) \circ x =$

$= ac.ba$ for all $a,b,c \in Q$.

Let Q be a WAD-quasigroup. A tetrad $(Q(\circ),g,h,x)$ is called an arithmetical form of Q if the condition (iv) from Theorem 1 is satisfied.

The following lemma is implicitly proved in [1], Theorem 4.9.

<u>Lemma 2</u>. Let Q be a WAD-quasigroup and $x \in Q$. Then there exists an arithemtical form $(Q(\circ),g,h,y)$ of Q such that $xx.xx = j$.

<u>Lemma 3</u>. Let Q be a WAD-quasigroup with an arithmetical form $(Q(\circ),g,h,x)$. Then $x \in N(Q(\circ))$ and $a \circ g(a)$, $a \circ h(a) \in N(Q(\circ))$ for every $a \in Q$, provided at least one of the following conditions holds:

(i) Q is an LF-quasigroup.

(ii) Q is an RF-quasigroup.

(iii) $(a.aa)(bc) = (ab)(aa.c)$ for all $a,b,c \in Q$.

(iv) $(bc)(aa.a) = (b.aa)(ca)$ for all $a,b,c \in Q$.

Proof. (i) As it is easy to see, $he(a) =$
$= (a \circ x^{-1}) \circ g(a^{-1})$ for each $a \in Q$. Since Q is an LF-quasigroup,

$$(g(a) \circ ((hg(b) \circ h^2(c)) \circ h(x))) \circ x = a.bc = ab.e(a)c =$$
$$= (((g^2(a) \circ gh(b)) \circ g(x)) \circ ((((g(a) \circ g(x^{-1})) \circ g^2(a^{-1})) \circ$$
$$\circ h^2(x)) \circ h(c))) \circ x,$$

and hence

$$(1) \quad a \circ ((b \circ c) \circ h(x)) = ((g(a) \circ b) \circ g(x)) \circ ((((a \circ g(x^{-1})) \circ$$
$$\circ g(a^{-1})) \circ c) \circ h(x))$$

for all $a,b,c, \in Q$. If we substitute $a = j$ in (1), we obtain the equality

$$h(x) \circ (b \circ c) = (b \circ g(x)) \circ ((c \circ g(x^{-1})) \circ h(x)).$$

Multiplying the last equality by $h(x^{-1}) \circ g(x)$ and taking into account that this element belongs to $N(Q(\circ))$ (since $gh =$ $= hg$ and gh^{-1} is nuclear), we get the equality $(b \circ c) \circ g(x) =$ $= (b \circ g(x)) \circ c$ for all $b,c \in Q$. Thus $g(x) \in N(Q(\circ))$, and consequently $h(x)$, $x \in N(Q(\circ))$. Now the equality (1) yields
$a \circ (b \circ c) = (g(a) \circ b) \circ ((a \circ g(a^{-1})) \circ c)$ and
$(a \circ b) \circ ((a^{-1} \circ g^{-1}(a)) \circ c) = g^{-1}(a) \circ (b \circ c) =$
$= (a \circ (a^{-1} \circ g^{-1}(a)) \circ (b \circ c)$ for all $a,b,c \in Q$. By Lemma 1, $a^{-1} \circ$ $a^{-1} \circ g^{-1}(a) \in N(Q(\circ))$. However $a \circ a \circ a \in N(Q(\circ))$ (since $Q(\circ)$ is a commutative Moufang loop), so that $a \circ g^{-1}(a) \in N(Q(\circ))$. As $N(Q(\circ))$ is invariant under automorphisms, $a \circ g(a) \in$ $\in N(Q(\circ))$. Finally, $a^{-1} \circ gh^{-1}(a)$ is contained in $N(Q(\circ))$, and therefore $g(a) \circ gh^{-1}(a) \in N(Q(\circ))$, $a \circ h^{-1}(a) \in N(Q(\circ))$ and $a \circ h(a) \in N(Q(\circ))$.

(ii) Similarly as for (i).

(iii) After some arrangements (using Lemma 1 and the fact that gh^{-1} is nuclear), we can write the identity

$(a.aa)(bc) = (ab)(aa.c)$ as $(a \circ h((a \circ hg^{-1}(a)) \circ g(x))) \circ$
$\circ (b \circ c) = (a \circ b) \circ (h((a \circ hg^{-1}(a)) \circ g(x)) \circ c)$.
If $a = j$ then $hg(x) \circ (b \circ c) = b \circ (hg(x) \circ c)$, and hence $x \in N(Q(\circ))$. Then $(a \circ h(a) \circ hg^{-1}(a))) \circ (b \circ c) =$
$= (a \circ b) \circ ((h(a \circ hg^{-1}(a) \ \circ c)$ and $a^{-1} \circ h(a \circ hg^{-1}(a)) \in$
$\in N(Q(\circ))$ by Lemma 1. However, gh^{-1} is nuclear, therefore hg^{-1} is so and $h(a \circ hg^{-1}(a^{-1})) \in N(Q(\circ))$. Thus $a^{-1} \circ h(a \circ a) \in N(Q(\circ))$. Finally, $h(a^{-1} \circ a^{-1} \circ a^{-1}) \in N(Q(\circ))$, so that $a \circ h(a) \in N(Q(\circ))$. Similarly as in the proof of (i), we can show that $a \circ g(a) \in N(Q(\circ))$.

(iv) Similarly as for (iii).

Lemma 4. Let a WAD-quasigroup Q have an arithmetical form $(Q(\circ), g, h, x)$ such that $x \in N(Q(\circ))$ and $a \circ g(a)$, $a \circ h(a) \in N(Q(\circ))$ for every $a \in Q$. Then

(i) Q is an F-quasigroup.

(ii) If $a,b,c,d \in Q$ and $ab.cd = ac.bd$ then $ab.(c(dd.dd)) = ac.(b(dd.dd))$.

Proof. (i) It is an easy exercise.

(ii) Since $ab.cd = ac.bd$ and $x \in N(Q(\circ))$,

(2) $(g^2(a) \circ gh(b)) \circ (hg(c) \circ h^2(d)) = (g^2(a) \circ gh(c)) \circ$
$\circ (hg(b) \circ h^2(d))$.

Put $u = (g^2(d) \circ h^2(d)) \circ (hg(d) \circ hg(d))$. We shall prove that $d^{-1} \circ u \in N(Q(\circ))$. Indeed, $d^{-1} \circ g(d^{-1})$, $g(d) \circ gh(d)$, $g(d) \circ g^2(d)$, $d^{-1} \circ h(d^{-1})$ and $h(d) \circ h^2(d)$ belong to $N(Q(\circ))$. Hence $d^{-1} \circ g^2(d)$, $d^{-1} \circ gh(d)$ and $d^{-1} \circ h^2(d)$ are contained in $N(Q(\circ))$. Since $d \circ d \circ d \in N(Q(\circ))$, $d \circ (gh(d) \circ gh(d)) \in$
$\in N(Q(\circ))$. However, $d^{-1} \circ ((d \circ d) \circ (gh(d) \circ gh(d))) =$
$= d \circ (gh(d) \circ gh(d))$ by the diassociativity of $Q(\circ)$ and

$d^{-1}\circ g^2(d)$, $d^{-1}\circ h^2(d)\in N(Q(\circ))$. Thus $d^{-1}\circ u\in N(Q(\circ))$, and so $h^2(d^{-1}\circ u)\in N(Q(\circ))$. Multiplying (2) by $h^2(d^{-1}\circ u)$, we obtain the equality

$(g^2(a)\circ gh(b))\circ(hg(c)\circ h^2(u)) = (g^2(a)\circ gh(c))\circ(hg(b)\circ h^2(u))$, and it is not so difficult to see that $ab.c(c(dd.dd)) =$
$= ac.(b(dd.dd))$.

Lemma 5. Let Q be an LF-quasigroup (RF-quasigroup) and $x,a,b\in Q$. Then

(i) $ef(x) = fe(x)$,

(ii) $L_bR_a = R_aL_b$ iff $e(b) = f(a)$.

Proof. (i) $x(ef(x).e(x)) = f(x)x.ef(x)\ e(x) =$
$= f(x).xe(x) = x = xe(x)$.

(ii) If $L_bR_a = R_aL$ then $ba = R_aL_b(e(b)) = L_bR_a(e(b)) =$
$= b.e(b)a$. Conversely, if $e(b) = f(a)$ then $b.ya = by.e(b)a =$
$= by.a$ for each $y\in Q$.

Lemma 6. Let Q be an F-quasigroup and $a,b\in Q$. Suppose that $L_bR_a = R_aL_b$ and $R_a^{-1}(x).L_b^{-1}(y) = R_a^{-1}(y).L_b^{-1}(x)$ for all $x,y\in Q$. Then Q is a WAD-quasigroup.

Proof. Put $x\circ y = R_a^{-1}(x).L_b^{-1}(y)$. Clearly, $Q(\circ)$ is a commutative loop. Let $k(x) = R_aR_{f(a)}R_a^{-1}(x)$ and $t(x) =$
$= L_bL_{e(b)}L_b^{-1}(x)$ for every $x\in Q$. As it is easy to see,
$R_a(x\circ y) = R_{f(a)}R_a^{-1}(x).R_aL_b^{-1}(y) = k(x)\circ R_a(y)$ and $L_b(x\circ y) =$
$= L_b(x)\circ t(y)$ for all $x,y\in Q$. Hence $k(x\circ y)\circ R_a(j) =$
$= k(x)\circ(k(y)\circ R_a(j))$ and $L_b(j)\circ t(x\circ y) = (L_b(j)\circ t(x))\circ t(y)$.
Now we can write

$R_a(x)\circ(L_bR_a(y)\circ tL_b(z)) = R_a(x)\circ L_b(R_a(y)\circ L_b(z)) = x.yz =$
$= xy.e(x)z = R_a(R_a(x)\circ L_b(y))\circ L_b(R_a(e(x))\circ L_b(z)) =$
$= (kR_a(x)\circ R_aL_b(y))\circ(L_bR_ae(x)\circ tL_b(z))$

for all $x,y,z \in Q$. Hence

$$x \circ (y \circ z) = (k(x) \circ y) \circ (L_b R_a e(R_a^{-1}(x)) \circ z),$$

$$(x \circ y) \circ (L_b R_a e R_a^{-1} k^{-1}(x) \circ z) = k^{-1}(x) \circ (y \circ z),$$

$$k^{-1}(x) = x \circ L_b R_a e R_a^{-1} k^{-1}(x)$$

for all $x,y,z \in Q$. According to Lemma 1, $Q(\circ)$ is a commutative Moufang loop, $L_b R_a e R_a^{-1} k^{-1}(x) = x^{-1} \circ k^{-1}(x)$ and $x^{-1} \circ x^{-1} \circ k^{-1}(x) \in N(Q(\circ))$. Therefore $x \circ k^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$. Similarly we can prove that $x \circ t^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$. Further, $k(x) \circ R_a(y) = R_a(x \circ y) = R_a(y \circ x) = k(y) \circ R_a(x)$, $x \circ R_a k^{-1}(y) = y \circ R_a k^{-1}(x)$ and $R_a(j) = k(j) \circ R_a(j)$. Hence $k(j) = j$, $R_a k^{-1}$ is a middle regular permutation of $Q(\circ)$ and $R_a(j) = R_a k^{-1}(j) \in N(Q(\circ))$. Similarly, $L_b(j) \in N(Q(\circ))$. Now it is obvious that both k and t are automorphisms of $Q(\circ)$ and $xy = (k(x) \circ t(y)) \circ (R_a(j) \circ L_b(j))$ for all $x,y \in Q$. Since $x^{-1} \circ k(x^{-1})$, $k(x) \circ kt^{-1}(x) \in N(Q(\circ))$ for every $x \in Q$, kt^{-1} is a nuclear mapping. Finally, $tk(x) \circ tR_a(j) \circ L_b(j) = tR_a(x) \circ L_b(j) = L_b R_a(x) = R_a L_b(x) = kt(x) \circ kL_b(j) \circ R_a(j)$. Thus $tR_a(j) \circ L_b(j) = kL_b(j) \circ R_a(j)$ and $tk = kt$. An application of Theorem 1 completes the proof.

Theorem 2. The following conditions are equivalent for every quasigroup Q:

(i) Q is a WAD-quasigroup and Q is an LF-quasigroup.

(ii) Q is a WAD-quasigroup and Q is an RF-quasigroup.

(iii) Q is a WAD-quasigroup and Q is an F-quasigroup.

(iv) Q is a WAD-quasigroup and $(a.aa)(bc) = (ab)(aa.c)$ for all $a,b,c \in Q$.

(v) Q is a WAD-quasigroup and $(bc)(aa.a) = (b.aa)(ca)$ for all $a,b,c \in Q$.

(vi) There are a commutative Moufang loop $Q(\circ)$, $g,h \in \in \mathrm{Aut}\, Q(\circ)$ and $x \in N(Q(\circ))$ such that $gh = hg$, $a \circ g(a)$, $a \circ h(a) \in N(Q(\circ))$ and $ab = (g(a) \circ h(b)) \circ x$ for all $a,b \in Q$.

(vii) If $a,b,c,d \in Q$ and $ab.cd = ac.bd$, then the subquasigroup generated by these elements is abelian.

(viii) Q is a triabelian quasigroup.

(ix) Every subgroupoid of Q which is generated by at most three elements is abelian.

(x) Q is an F-quasigroup and there exists $z \in Q$ such that $f(z)a.be^2(z) = f(z)b.ae^2(z)$ for all $a,b \in Q$.

(xi) Q is an F-quasigroup and there exists $z \in Q$ such that $f^2(z)a.be(z) = f^2(z)b.ae(z)$ for all $a,b \in Q$.

<u>Proof</u>. The implications (i) implies (vi), (ii) implies (vi), (iii) implies (vi), (iv) implies (vi) and (v) implies (vi) follow from Lemma 3 and Theorem 1.

(vi) implies (vii). As it is easy to see, gh^{-1} is a nuclear mapping. By Theorem 1 and Lemma 4, Q is an F-quasigroup and a WAD-quasigroup. With respect to Lemma 2 and Lemma 3, we may assume that $j = dd.dd$. Then (by Lemma 4(ii)) $ab.cj = = ac.bj$ and $(g^2(a) \circ gh(b)) \circ gh(c) = (g^2(a) \circ gh(c)) \circ gh(b)$. Let $G(\circ)$ be the subloop of $Q(\circ)$ generated by $N(Q(\circ)) \cup \cup \{g^2(a), gh(b), gh(c)\}$. According to the well-known Moufang theorem, $G(\circ)$ is an abelian group. Since $x \in N(Q(\circ))$ and $z \circ g(z)$, $z \circ h(z)$, $z \circ g^{-1}(z)$, $z \circ h^{-1}(z) \in N(Q(\circ))$, $g(G) = = h(G) = G$, G is an abelian subquasigroup in Q and $a,b,c \in G$. Finally, $(g^2(d) \circ gh(d)) \circ (gh(d) \circ h^2(d)) = x^{-1} \circ g(x^{-1}) \circ h(x^{-1}) \in \in N(Q(\circ))$ and $g^2(d^{-1}) \circ d^{-1}$, $h^2(d^{-1}) \circ d^{-1}$, $gh(d^{-1}) \circ d^{-1}$, $d \circ d \circ d \in N(Q(\circ))$. Thus $d^{-1} \in N(Q(\circ))$ and $d \in G$.

(vii) implies (viii). This implication is obvious, since ab.bc = ab.bc. The implications (viii) implies (ix) and (ix) implies (iv), (v) are trivial and the implication (vi) implies (i), (ii), (iii) follows from Theorem 1 and Lemma 4(i).

(x) implies (i). Put $x = fe(z)$ and $y = e^2(z)$. By Lemma 5, $e(x) = f(y)$ and $R_y L_x = L_x R_y$. According to the hypothesis, $f(z)R_y^{-1}(a).b = f(z)R_y^{-1}(b).a$ for all $a,b \in Q$. Hence $R_y^{-1}(a).L_x^{-1}(b) = L_{f(z)}^{-1}(f(z)R_y^{-1}(a).b) = R_y^{-1}(b).L_x^{-1}(a)$ and we can use Lemma 6.

Similarly we can prove that (xi) implies (i).

The remaining implication (viii) implies (x),(xi) is trivial.

Corollary 1. A quasigroup Q is triabelian iff it satisfies the identity $((aa.bc)(xy.zz))((uv.wu)((r.rr)(st))) = ((ab.ac)(xz.yz))((uw.vu)((rs)(rr.t)))$ for all $a,b,c,x,y,z,u,v,w,r,s,t \in Q$.

Corollary 2. Triabelian quasigroups are finitely based.

Corollary 3. Every commutative F-quasigroup is triabelian.

Proof. Let Q be a commutative F-quasigroup and $z \in Q$. Then $f(z)a.be^2(z) = e(z)a.e^2(z)b = e(z).ab = e(z).ba = f(z)b.ae^2(z)$ for all $a,b \in Q$.

Corollary 4. Every triabelian quasigroup is isotopic to a totally symmetric triabelian quasigroup with at least one idempotent element.

Corollary 5. Every totally symmetric quasigroup isotopic to a Moufang loop is triabelian.

Proof. Let Q be a totally symmetric quasigroup isotopic to a Moufang loop, $z \in Q$, $g = L_z$ and $a \circ b = g(a).g(b)$ for all $a,b \in Q$. Then $Q(\circ)$ is a commutative Moufang loop ($Q(\circ)$ is clearly commutative and every loop isotopic to a Moufang loop is Moufang) and $h(a) \circ h(h(a) \circ h(b)) = b$ for all $a,b \in Q$ and $h = g^{-1}$. For $a = g(j)$ we obtain the equality $h^2(b) = b$. Hence $h(a) \circ h^2(a) = h(a) \circ a = h(a) \circ h(h(a) \circ h(y)) = y$ for all $a \in Q$ and $y = g(j)$. Thus $h(a) = y \circ a^{-1}$ and we can write $x \circ (a^{-1} \circ (x^{-1} \circ (a \circ b))) = a.ab = b$ for all $a,b \in Q$ and $x = y \circ y$. Now it is visible that $a^{-1} \circ (x^{-1} \circ (a \circ b)) = x^{-1} \circ (a^{-1} \circ (a \circ b))$ and $x^{-1} \in N(Q(\circ))$. Consequently $x \in N(Q(\circ))$ and we can use Theorem 2(vi).

Let Q be a quasigroup. A mapping g of Q into Q is called left regular if there is a mapping h such that $g(xy) = h(x).y$ for all $x,y \in Q$.

Theorem 3. Let Q be a triabelian quasigroup. Define a binary relation r on Q by a r b iff $a = t(b)$ for some left regular mapping t. Then

(i) If $(Q(\circ), g,h,x)$ is an arithmetical form of Q and $a, b \in Q$ then a r b iff $b = a \circ y$ for some $y \in N(Q(\circ))$.

(ii) r is a normal congruence relation of Q.

(iii) The factorquasigroup Q/r is an idempotent totally symmetric triabelian quasigroup.

(iv) The set $\{z \mid z \ r \ a\}$ is an abelian subquasigroup in Q for every $a \in Q$.

Proof. (i) Let $a = t(b)$ for a left regular mapping t. Then there is a mapping s such that $t((g(c) \circ h(d) \circ x) = (gs(c) \circ h(d)) \circ x$ for all $c,d \in Q$. Substituting $h^{-1}(x^{-1})$ for d we obtain the equality $tg(c) = gs(c)$. Hence $t((c \circ d) \circ x) =$

$= (t(c) \circ d) \circ x$ for all $c,d \in Q$, so that $t(c \circ x) = t(c) \circ x$. Consequently $t(d) = t(j) \circ d$ and $t(j) \in N(Q(\circ))$. The rest is obvious.

(ii) can be proved easily using (i).

(iii) Let $(Q(\circ),g,h,x)$ be an arithmetical form of Q and $a \in Q$. Then (Lemma 3) $a \circ g(a)$, $a \circ h(a)$, $x \in N(Q(\circ))$. Hence the elements $a^{-3} \circ ((a \circ g(a)) \circ (a \circ h(a))) =$
$= a^{-3} \circ ((a \circ a) \circ (g(a) \circ g(a))) = a^{-1} \circ (g(a) \circ h(a))$ and $k(a) =$
$= a^{-1} \circ ((g(a) \circ h(a)) \circ x)$ belong to $N(Q(\circ))$. Further, $a \circ k(a) = aa$, $a = aa \circ (k(a))^{-1}$ and $a \; r \; aa$ by (i). Thus Q/r is an idempotent quasigroup. The rest is an easy consequence of the fact that $a \circ a \circ a \in N(Q(\circ))$ for every $a \in Q$.

(iv) follows from (i), (iii) and Lemma 2.

Corollary 6. Every simple triabelian quasigroup is abelian.

Proof. Let Q be a simple triabelian quasigroup with an arithmetical form $(Q(\circ),g,h,x)$. Consider the normal congruence relation r defined in Theorem 3. If $r = Q \times Q$, then Q is abelian by Theorem 3(iv). Let $r \neq Q \times Q$. Then r is the identical relation (since Q is simple) and Q is idempotent and totally symmetric as it follows from Theorem 3(iii). In this case, $g(a) = h(a) = a^{-1}$ for every a and $x = j$. It is easy to see that every congruence of $Q(\circ)$ is a congruence of Q, and consequently $Q(\circ)$ is simple. However, every simple commutative Moufang loop is a group.

References

[1] KEPKA T.: Quasigroups which satisfy certain generalized forms of the abelian identity, Čas. Pěst. Mat. 100,

(1975),46-60.

[2] ORLIK - PFLUGFELDER H.: A special class of Moufang loops, Proc. Amer. Math. Soc. 26(1970),583-586.

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

(Oblatum 23.4.1975)