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ON CONTINUOUS IMAGES OF EBERLEIN COMPACTS

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Abstract: J. Lindenstrauss ([L]) has raised the question, whether each Hausdorff continuous image of an Eberlein compact is an Eberlein compact again. The aim of the present paper is to prove that the answer is affirmative in two particular cases. Since the nature of the problem is purely topological, nothing about the relationship with the theory of Banach spaces will be mentioned, the reader is recommended to [L] and [AA], where he can find also further references.

Key words and phrases: Eberlein compact, dispersed space, point-finite collection, weakly separating collection.

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0. Conventions and notations. In the whole paper, the word "space" will mean "topological Hausdorff space", similarly, "continuous image of a space X" will be a Hausdorff space Y which is an image of X under a continuous onto mapping. The closed unit interval $[0,1]$ will be denoted by I, the two-point set (and the discrete two-point topological space) will be denoted by 2 and its elements by 0 and 1. The symbols Open (X) (resp. Clopen(X), resp. Coz(X)) denote the set of all open (resp. clopen, resp. cozero) subsets of a space X.

1. Definition ([L]). A compact space X will be cal-

led an Eberlein compact, if there exists an embedding of X into some cube I^Γ such that for every $x \in X$ and for every real $r > 0$ the set of all indices $\gamma \in \Gamma$ with $x(\gamma) > r$ is finite.

H.P. Rosenthal has proved that Eberlein compacts can be characterized using the special covering property:

<u>2. Proposition</u> ([R1]). A compact space X is an Eberlein compact if and only if there exists a σ-point finite system $\mathcal{C} \subset Coz(X)$ which weakly separates points of X, i.e. for any two different $x, y \in X$ there is a $C \in \mathcal{C}$ such that $\{x,y\} \cap C \neq \emptyset$ and $\{x,y\} - C \neq \emptyset$.

Strengthening the condition in Proposition 2 one obtains the following definition:

<u>3. Definition</u>. A compact space X will be called a strong Eberlein compact, if there exists a point-finite system $\mathcal{C} \subset Coz(X)$, weakly separating points of X.

<u>4. Convention</u>. When dealing with an Eberlein compact X, which is embedded into I^Γ (resp. into 2^Γ), we shall always assume that the embedding satisfies the condition described in Definition 1 (resp. in Proposition 8), i.e. that for each $x \in X$ is true that $\operatorname{card}\{\gamma \in \Gamma \mid x(\gamma) > r\} < \omega_0$ for each $r > 0$ (resp. $\operatorname{card}\{\gamma \in \Gamma \mid x(\gamma) = 1\} < \omega_0$).

We want to prove that the continuous image of a strong Eberlein compact is again a strong Eberlein compact. To this end, we need some propositions. Let us recall that a space X is called to be dispersed, if X does not contain a non-void perfect subspace, that means, each subspace Z of X has at least one point isolated in Z.

5. Proposition. Every strong Eberlein compact is dispersed.

Proof. Suppose the contrary, let $Z \subset X$ be a non-void perfect closed subspace of X, let $\mathcal{C} \subset \mathrm{Coz}(X)$ be the point-finite system weakly separating points of X. Let us choose a point $x_1 \in Z$. The set $C_1 = \bigcap \{ C \in \mathcal{C} \mid x_1 \in C \}$ is a cozero set by point-finiteness of $\mathcal{C}$, let F_1 be some closed perfect set, such that $x_1 \in F_1 \subset C_1 \cap Z$. Since F_1 is compact and perfect, there is some point x_2 in F_1 different from x_1, let $C_2 = \bigcap \{ C \in \mathcal{C} \mid x_2 \in C \}$. $\mathcal{C}$ weakly separates points, thus $x_1 \notin C_2$. Let us choose closed perfect $F_2 \subset C_2 \cap F_1$ with $x_2 \in F_2$; proceeding by the obvious induction we can find a strictly decreasing sequence of closed non-empty subsets $\{F_n\}$. The space X is compact, thus the intersection $\bigcap \{F_n\}$ is non-void, each point of this intersection belongs to infinitely many C's - a contradiction with point-finiteness of $\mathcal{C}$.

6. Proposition. Every non-void compact dispersed space is 0-dimensional.

Proof. We must show that each point has arbitrarily small clopen neighborhood. Let X be the space in question, since X is dispersed, one can write $X = \bigcup \{ X_\iota \mid \iota < \alpha \}$ where X_ι is the set of all isolated points of $\bigcup \{ X_\varkappa \mid \iota \leq \varkappa < \alpha \}$, and α is suitable ordinal.

The proof goes by transfinite induction. For $\iota = 0$, let $x \in X_0$. It means that x is isolated in X, thus $\{x\}$ is the clopen neighborhood of x.

Let $\iota < \alpha$ and let for every $\varkappa < \iota$ and for every $y \in X_\varkappa$ be true that y has a clopen neighborhood base. Let U be a neighborhood of x, $x \in X_\iota$. Since x is isolated in $\bigcup \{X_\mu \mid \iota \le \mu < \alpha\}$ and since X is regular, there is some neighborhood V of x with the following properties: $\mathrm{cl} V \subset U$, $\mathrm{cl}\, V \cap \bigcup \{X_\mu \mid \iota \le \mu < \alpha\} = \{x\}$. Thus the boundary $\mathrm{bd}\, V$ is contained in $\bigcup \{X_\varkappa \mid \varkappa < \iota\}$.
For a point $y \in \mathrm{bd}\, V$ let W_y be a clopen neighborhood of y which does not contain x (by the induction hypothesis such a neighborhood does exist). The boundary $\mathrm{bd}\, V$ is compact and so some finite collection $\{W_{y_1}, W_{y_2}, \dots, W_{y_n}\}$ covers it. It is self-evident that the set $V - \bigcup \{W_{y_i} \mid i = 1,2,\dots,n\}$ is the clopen neighborhood of x, contained in U.

<u>7. Proposition</u>. Let X be 0-dimensional (resp. strong) Eberlein compact. Then there exists a σ-point-finite (resp. point-finite) system $\mathcal{D} \subset \mathrm{Clopen}(X)$, which weakly separates points.

The proof is routine. One needs only to realize that every cozero set in 0-dimensional compact space is a union of a point-finite collection of clopen sets.

<u>8. Proposition</u>. Every strong Eberlein compact can be embedded into 2^Γ for some set of indices Γ in such a manner that for every $x \in X$ the set $\{\gamma \in \Gamma \mid x(\gamma) = 1\}$ is finite.

Proof. Let $\mathscr{C} \subset \text{Clopen}(X)$ be a point-finite system, which weakly separates points of X. For $C \in \mathscr{C}$ let f_C: $: X \longrightarrow 2$ be the mapping which maps C onto 1, $X - C$ onto 0. Let $\psi : X \longrightarrow 2^{\mathscr{C}}$ be defined by the rule $\psi(x)(C) = f_C(x)$. Then the mapping ψ is the desired embedding. (Easy.)

<u>9. Proposition</u>. Let X be compact space and let there exist a point finite system $\mathcal{O} \subset \text{Open}(X)$, which weakly separates points of X. Then X is a strong Eberlein compact.

Proof. For $x \in X$, denote $O_x = \bigcap \{ O \in \mathcal{O} \mid x \in O \}$, let C_x be a non-empty cozero set with $x \in C_x \subset O_x$. Then $\mathscr{C} = \{ C_x \mid x \in X \} \subset \text{Coz}(X)$ is a point-finite system weakly separating points of X.

<u>Remark</u>. Proposition 9 cannot be generalized to the case of a σ-point-finite collection and general Eberlein compact, as the following example shows: For every $x \in I$ denote by $\{ q_n(x) \}$ some sequence of rational numbers in I, converging to x in usual topology. Let $Q = \{ q_n(x) \mid x \in I, n \in \omega_0 \}$. Let Y be the space, whose underlying set is a disjoint union of I and Q, and whose topology is defined as follows: each $q \in Q$ is isolated, and the neighborhood base of $x \in I$ has members of form $\{ q_k(x), q_{k+1}(x), q_{k+2}(x), \dots, x \}$, with k natural. Let X be the one-point compactification of Y. Then: X is not an Eberlein compact, but X admits a σ-point-finite collection of open sets, which weakly separates points. The verification of both properties may be left to the reader.

10. Definition. Let $X \subset 2^{\Gamma}$ be a strong Eberlein compact, let Y be its continuous image under the mapping f. For $x \in X$ let us define

$$dg\, x = \operatorname{card} \{ \gamma \in \Gamma \mid x(\gamma) = 1 \}$$

and for $y \in Y$ define

$$dg\, y = \min \{ dg\, x \mid x \in f^{-1}(y) \} .$$

11. Lemma: Let $X \subset 2^{\Gamma}$ be a strong Eberlein compact, $x \in X$, and $\{x_n\}$ a sequence of points converging to x, $x_n \neq x$ for all n. Then $dg\, x_n > dg\, x$ holds for all but finitely many n.

The proof is easy - the set $\{ y \in X \mid x(\gamma) = 1 \Longrightarrow \Longrightarrow y(\gamma) = 1 \}$ is a neighborhood of x and thus contains almost all x_n.

12. Lemma. Let $X \subset 2^{\Gamma}$ be a strong Eberlein compact, Y its continuous image under the mapping f, $y_0 \in Y$. Then the set

$$\bigcup \{ f^{-1}(y) \mid y \in Y , \ dg\, y \leq dg\, y_0 , \ y \neq y_0 \}$$

is closed.

Proof. Let $dg\, y_0 = n$; denote by M the set above. Pick a point $x \in \operatorname{cl} M$. Since each Eberlein compact is a Fréchet space, there is a sequence $\{x_n\}$ ranging in M, which converges to x. If infinitely many members of $\{x_n\}$ belong to some $f^{-1}(y)$, then $x \in f^{-1}(y)$ and so $x \in M$.

In other case we may assume without loss of generality that $x_n \in f^{-1}(y_n)$, all y_n being distinct, and that the sequence $\{y_n\}$ converges to $y = f(x)$. Let us choose a point $t_n \in f^{-1}(y_n)$ with $dg\ t_n = dg\ y_n$. Again we may assume that the sequence $\{t_n\}$ converges; denote by t its limit point. Since $dg\ t_n \leq n$, when applying Lemma 11 we obtain that $dg\ t < n$. It follows that $x \in \bigcup \{ f^{-1}(y) \mid y \in Y ,$ $dg\ y < dg\ y_0 \} \subset M$, thus, x having been chosen from cl M arbitrarily, cl $M \subset M$.

<u>13. Theorem.</u> A continuous image of a strong Eberlein compact is a strong Eberlein compact, too.

Proof. Suppose $X \subset 2^{\Gamma}$ be a strong Eberlein compact, f a continuous mapping from X onto Y . Denote by O_x the set $\{y \in X \mid x(\gamma) = 1 \Longrightarrow y(\gamma) = 1\}$, let $\mathcal{O} = \{ O_x \mid x \in X \}$. The system $\mathcal{O}$ is point-finite, consists of clopen sets and weakly separates points.

For a point $x \in X$ the set $\bigcup \{ f^{-1}(y) \mid dg\ y \leq dg\ f(x) ,$ $y \neq f(x) \}$ is closed by Lemma 12, and the point x is not its member. Thus we can find a neighborhood V_x of x , disjoint with the set $\bigcup \{ f^{-1}(y) \mid dg\ y \leq dg\ f(x) , \ y \neq f(x) \}$.
Let us denote $U_x = O_x \cap V_x$, $\mathcal{U} = \{ U_x \mid x \in X \}$. The collection $\mathcal{U}$ is point-finite and weakly separating, because the collection $\mathcal{O}$ is.

For every $y \in Y$ let us choose a finite family $\{U_{x_1}, U_{x_2}, \ldots, U_{x_{n(y)}}\} \subset \mathcal{U}$ such that $x_i \in f^{-1}(y)$ and $\bigcup \{ U_{x_i} \mid i = 1,2,\ldots,n(y)\} \supset f^{-1}(y)$. Let us denote the last union by $U(f^{-1}(y))$.

a) The system $\{U(f^{-1}(y)) \mid y \in Y\}$ is point-finite:
If a point x belongs to infinitely many $U(f^{-1}(y))$'s, then

by the definition of $U(f^{-1}(y))$ it must belong to infinitely many different U_{x_i}'s, which is impossible, because $\mathcal{U}$ is point-finite.

b) For any two $y_1, y_2 \in Y$, $y_1 \neq y_2$, either $U(f^{-1}(y_1)) \cap f^{-1}(y_2) = \emptyset$ or $f^{-1}(y_1) \cap U(f^{-1}(y_2)) = \emptyset$:

Suppose $\operatorname{dg} y_1 \geq \operatorname{dg} y_2$. Then for all $x \in f^{-1}(y_1)$, $V_x \cap (\bigcup \{ f^{-1}(y) \mid \operatorname{dg} y \leq \operatorname{dg} y_1 , y \neq y_1 \} = \emptyset$, hence $U_x \cap f^{-1}(y_2) = \emptyset$. Thus $\bigcup \{ U_x \mid x \in f^{-1}(y_1) \} \cap f^{-1}(y_2) = \emptyset$, consequently $U(f^{-1}(y_1)) \cap f^{-1}(y_2) = \emptyset$.

Now it suffices to define for each $y \in Y$ the set C_y by the equality $C_y = Y - f[X - U(f^{-1}(y))]$ and to denote $\mathcal{C} = \{ C_y \mid y \in Y \}$. $\mathcal{C}$ consists of open sets, a) implies its point-finiteness, according to b) $\mathcal{C}$ weakly separates points of Y, and it remains to apply Proposition 9 to obtain that Y is a strong Eberlein compact.

There is another special case, when I can prove that a continuous image of an Eberlein compact is an Eberlein compact. I am very sorry that the proof of the following theorem is very technical and long, but I don't know any better.

__14. Theorem.__ Let X be an Eberlein compact, $K \subset X$ its closed subset. Then the quotient X/K is an Eberlein compact.

Proof. By Proposition 2 it will suffice, if one can find in X some σ-point-finite collection $\mathcal{C} \subset \operatorname{Coz}(X)$ with the following properties:

a) $\mathcal{C}$ weakly separates points of $X - K$

b) $\bigcup \mathcal{C}$ is disjoint with K

c) $\mathcal{C}$ covers $X - K$.

We may assume that X is embedded into I^Γ and the embedding satisfies Definition 1, and that the point $\underline{0}$ (= the point, whose all coordinates equal to zero) belongs to the set K . The sets of the form $D_{\gamma,j,n} =$
$= \pi_\gamma^{-1}[\,]^{j}/_{n}, {}^{j+2}/_{n}[\,] \cap X$ (where π_γ is the γ-th projection) are cozero sets and it is easy to check that the system $\{D_{\gamma,j,n} \mid \gamma \in \Gamma,\ n \in \omega_0,\ 1 \leq j \leq n-1\}$ is - point-finite, weakly separates points of X and covers $X - \{\underline{0}\}$. Fix n for a moment, and let $\mathcal{D}_n$ be a collection of all intersections of finitely many $D_{\gamma,j,n}$; $\mathcal{D}_n$ is point-finite and $\mathcal{D} = \bigcup\{\mathcal{D}_n \mid n \in \omega_0\}$ weakly separates points.

For $D \in \mathcal{D}_n$ let us introduce the following notation: If $F = \{\gamma_1, \gamma_2, \dots, \gamma_k\} \subset \Gamma$, $p = \{j_{\gamma_1}, j_{\gamma_2}, \dots, j_{\gamma_k}\}$ finite collection of natural numbers satisfying the inequality
$1 \leq j_{\gamma_i} \leq n-1$ for all i , and if $D' = \bigcap\{D_{\gamma_i, j_i, n} \mid i = 1,2,\dots,k\}$, then let us denote $D = D^n_{F,p}$.

Call a set G of indices to be remarkable with respect to $D^n_{F,p}$, if

(i) $G \cap F = \emptyset$

(ii) there exists a point $x_G \in \mathrm{cl}\, D^n_{F,p} \cap K$ such that $x_G(\gamma) > {}^2/_n$ for all $\gamma \in G$, and $x_G(\gamma) \leq {}^2/_n$ for all $\gamma \in \Gamma - (F \quad G)$,

(iii) if there is some $y \in \mathrm{cl}\, D^n_{F,p} \cap K$ such that the set $H = \{\gamma \in \Gamma - F \mid y(\gamma) > {}^2/_n\}$ is contained in G , then

$H = G$.

(The case $G = \emptyset$ is not excluded.)

Now, some more notation.

$\mathcal{M}(D^n_{F,p})$ will be the set $\{G \subset \Gamma \mid G$ is remarkable with respect to $D^n_{F,p}\}$, and $L(D^n_{F,p}) = \bigcup \mathcal{M}(D^n_{F,p})$.

Lemma A. For every $D^n_{F,p}$, the set $\mathcal{M}(D^n_{F,p})$ is finite. Of course, as a corollary, the set $L(D^n_{F,p})$ is finite, too.

Put $C^n_{F,p} = D^n_{F,p} \cap \bigcap \{\sigma_\gamma^{-1}[[0, 2/m[] \mid \gamma \in L(D^m_{F,p})\}$. Denote

$\mathcal{C}_m = \{C^n_{F,p} \mid D^n_{F,p} \in \mathcal{D}_m \;, \; D^n_{F,p} \neq \emptyset \;, \; C^n_{F,p} \cap K = \emptyset\}$.

Lemma B. For every $x \in X - K$ and for each $D^n_{F,p}$ containing x , there exist n_1, F_1, p_1 such that $x \in C^{n_1}_{F_1,p_1} \subset D^n_{F,p}$ and $C^{n_1}_{F_1,p_1} \in \mathcal{C}_m$.

Then $\mathcal{C} = \bigcup\{\mathcal{C}_m \mid m \in \omega_0\}$ is the desired system, since its members are cozero sets, point-finiteness of $\mathcal{D}_m$ implies point-finiteness of $\mathcal{C}_m$ and thus $\mathcal{C}$ is σ-point-finite.

For $x \neq y$, both belonging to $X - K$, we can find a $D^n_{F,p}$ separating them. Suppose $x \in D^n_{F,p}$, $y \notin D^n_{F,p}$. According to Lemma B the set $C^{n_1}_{F_1,p_1}$ belongs to $\mathcal{C}$ and separates x and y .

All the members of $\mathcal{C}$ are disjoint with K - see the definition of $\mathcal{C}_m$.

Finally, since each point of $X - K$ is distinct from the origin $\underline{O}$, there must be some $D^n_{F,p}$ containing it. Again we may apply Lemma B to obtain that $\mathcal{C}$ covers $X - K$. Thus the conditions a), b), c) are verified and it remains to show the validity of both Lemmas, then the proof will be complete.

Proof of Lemma A. Suppose the contrary. Let $D^n_{F,p}$ belong to $\mathcal{D}_n$ and let for this particular $D^n_{F,p}$ the collection $\mathcal{M}(D^n_{F,p})$ (denote it by $\mathcal{M}$) be infinite. For every $G \in \mathcal{M}$ let us choose a point x_G having the properties from (ii). Since (by (iii)) the members of $\mathcal{M}$ are distinct, the set $\{x_G \mid G \in \mathcal{M}\}$ is infinite, let z be its accumulation point (X is compact!). The point z belongs to $\mathrm{cl}\, D^n_{F,p} \cap K$ because all x_G belong to this intersection. Let us denote $H = \{\gamma \in \Gamma - F \mid z(\gamma) > 2/n\}$. Choose real $r > 0$, $r < 1/2n$ and define an open neighborhood U of a point z by the following:

$$\pi_{\gamma_i}[U] = \left] j_{\gamma_i}/n - r \,,\, (j_{\gamma_i}+2)/n + r \right[\quad \text{for } \gamma_i \in F$$

$$\pi_\gamma[U] = \left] 2/n \,,\, 1 \right] \quad \text{for } \gamma \in H$$

$$\pi_\gamma[U] = [0,1] \quad \text{for } \gamma \in \Gamma - (F \cup H) .$$

By (iii) there is at most one x_G with $G = H$ and z is an accumulation point, so there must be some x_G with $G \neq H$, which belongs to U ; then obviously $G \supset H$. But by (iii) the sharp inclusion $G \supsetneqq H$ contradicts to the assumed remarkability of G . Thus $\mathcal{M}$ is finite. As a consequence (all members of $\mathcal{M}(D^n_{F,p})$ are finite), the set of indices $L(D^n_{F,p})$ is finite, too.

Proof of Lemma B. Pick a point $x \in X - K$ and a set $D^n_{F,p} \in \mathcal{D}_m$ which is a neighborhood of x.

Only two cases may occur:

1) There are n', F', p', such that $x \in D^{n'}_{F',p'} \subset D^n_{F,p}$ and $\mathrm{cl}\, D^{n'}_{F',p'} \cap K = \emptyset$.

Then the only set of indices remarkable with respect to $D^{n'}_{F',p'}$ is the empty set and thus $C^{n'}_{F',p'} = D^{n'}_{F',p'}$. It remains to write $n_1 = n'$, $F_1 = F'$, $p_1 = p'$.

2) For every triple n', F', p' satisfying $x \in D^{n'}_{F',p'} \subset$ $\subset D^n_{F,p}$ the set $\mathrm{cl}\, D^{n'}_{F',p'} \cap K$ is always non-void.

In this case, let Z be the intersection of all such $\mathrm{cl}\, D^{n'}_{F',p'}$. One can immediately observe that $Z = \{ y \in X \mid$ if $x(\gamma) \neq 0$, then $y(\gamma) = x(\gamma) \}$.

For $G \subset \Gamma - \{ \gamma \mid x(\gamma) > 0 \}$ and for real $r > 0$ let $W^r_G = \{ y \in X \mid \gamma \in G \Longrightarrow y(\gamma) \leq r \}$. Because $Z \cap \bigcap W^r_G =$ $= \{ x \}$ as may be easily checked (the intersection is taken over all $r > 0$ and all G finite, $G \subset \Gamma - \{ \gamma \mid x(\gamma) >$ $> 0 \}$), the intersection $Z \cap \bigcap W^r_G \cap K$ is empty. Since X is compact, W^r_G are closed, there exist some $r > 0$ and some finite $G \subset \{ \gamma \in \Gamma \mid x(\gamma) = 0 \}$ such that $Z \cap W^r_G \cap K = \emptyset$. The set Z is an intersection of a centered system: using the same argument, we obtain that there is some $D^{n_0}_{F_0,p_0}$ such that $\mathrm{cl}\, D^{n_0}_{F_0,p_0} \cap W^r_G \cap K = \emptyset$, and $x \in D^{n_0}_{F_0,p_0} \subset D^n_{F,p}$.

Let n_1 be a natural number, satisfying the following inequalities:

$1/m_1 < 1/2 \operatorname{Min}\{dist(x(\gamma), I - \pi_\gamma[D^{n_0}_{F_0,p_0}]) | \gamma \in F_0\}$,
$m_1 \geq n_0$, $1/m_1 < r/2$.

Denote $F_1 = \{\gamma | x(\gamma) > 1/m_1\}$. Then there is a finite sequence of natural numbers, p_1 , indexed by members of F_1 such that $x \in D^{n_1}_{F_1,p_1}$ and $D^{n_1}_{F_1,p_1} \subset D^{n_0}_{F_0,p_0}$. Construct $C^{n_1}_{F_1,p_1}$, obviously $x \in C^{n_1}_{F_1,p_1}$. We must show that $C^{n_1}_{F_1,p_1}$ belongs to $\mathcal{C}_{n_1}$; to this end it is sufficient to verify that $C^{n_1}_{F_1,p_1} \cap K = \emptyset$.

Before it, let us show that the empty set of indices is not remarkable with respect to $D^{n_1}_{F_1,p_1}$. Suppose the contrary. Then by (ii) there is a point $x_\emptyset$ belonging to $cl\ D^{n_1}_{F_1,p_1} \cap K$, for which $x_\emptyset(\gamma) \leq 2/m_1$ for all $\gamma \in \Gamma - F_1$, thus $x_\emptyset \in W^r_G$, because $G \subset \Gamma - F_1$ and $r > 2/m_1$. It follows that $x_\emptyset \in cl\ D^{n_1}_{F_1,p_1} \cap W^r_G \cap K \subset cl\ D^{n_0}_{F_0,p_0} \cap W^r_G \cap K = \emptyset$, a contradiction.

Thus $L(D^{n_1}_{F_1,p_1}) \neq \emptyset$.

We want to prove that $C^{n_1}_{F_1,p_1} \cap K = \emptyset$, suppose the contrary: let $y \in C^{n_1}_{F_1,p_1} \cap K$. From the previous we know that $cl\ D^{n_0}_{F_0,p_0} \cap W^r_G \cap K = \emptyset$, $C^{n_1}_{F_1,p_1} \subset D^{n_0}_{F_0,p_0}$ and $y \in C^{n_1}_{F_1,p_1} \cap K$: it follows that $y \notin W^r_G$, and consequently, the set of indices $H = \{\gamma \in \Gamma | y(\gamma) > 2/m_1\}$ is non-void. Now, since $y \in cl\ D^{n_1}_{F_1,p_1} \cap K$, we may use (iii), there is some $H_0 \subset H$, remarkable with respect to $D^{n_1}_{F_1,p_1}$. Because $\emptyset$ is not remarkable, $H_0 \neq \emptyset$. By the definition of $C^{n_1}_{F_1,p_1}$, for $\gamma \in L(D^{n_1}_{F_1,p_1})$ and for all $z \in C^{n_1}_{F_1,p_1}$ it is true that $z(\gamma) < 2/n_1$. But assuming $y \in C^{n_1}_{F_1,p_1}$ we see that $H_0 \cap L(D^{n_1}_{F_1,p_1}) = \emptyset$, which contradicts to the definition of $L(D^{n_1}_{F_1,p_1})$: H_0 is remarkable, disjoint with $L(D^{n_1}_{F_1,p_1})$, nevertheless $L(D^{n_1}_{F_1,p_1})$ was defined as the union of all remarkable sets.

The proof of Theorem 14 is complete.

Finally, the following theorem about the general case of Lindenstrauss' problem may - by the author's opinion - show that there is some relationship between Theorem 13 and the general case.

15. Theorem. The following statements are equivalent:

(a) Every continuous image of an Eberlein compact is an Eberlein compact.

(b) Every continuous image of a countable product of strong Eberlein compacts is an Eberlein compact.

Proof. (a) $\Longrightarrow$ (b) is obvious.
For the reverse implication, let X be an Eberlein compact, Y a compact space, $f: X \longrightarrow Y$ a continuous onto map. Suppose X to be embedded into some cube I^{Γ} . Since the Cantor discontinuum D can be mapped continuously onto I , there is a compact subset $Z \subset D^{\Gamma}$ and a mapping h from Z onto X , moreover h and Z can be defined in such a way that $Z =$

when considered with identical embedding D^Γ into I^Γ is again an Eberlein compact by Definition 1. Z is zero-dimensional, thus there is a family $\mathcal{C} = \bigcup \mathcal{C}_n \subset$ Clopen(Z) , which weakly separates points of Z , with each $\mathcal{C}_n$ point-finite (Proposition 7). Define an embedding of Z into $2^{\mathcal{C}}$ as in the proof of Proposition 8 and denote it ψ . Now, consider the projection $\pi_n : 2^{\mathcal{C}} \longrightarrow 2^{\mathcal{C}_n}$. Then $\pi_n [\psi [Z]] = Z_n$ is a strong Eberlein compact since $\mathcal{C}_n$ is point-finite, and, obviously, $\psi(Z) \subset \Pi\{Z_n | n \in \omega_0\}$.

One may easily define a map g and a compact Hausdorff space Y' , such that $Y \subset Y'$, $g[\Pi Z_n] = Y'$ and $g/\psi[Z] = f \circ g \circ \psi^{-1}$. Finally, if (b) holds, Y' is an Eberlein compact and Y , as a compact subspace of Y' , is an Eberlein compact, too.

Problem. Every strong Eberlein compact is dispersed. Is it true that each dispersed Eberlein compact is strong?

Added in proof. After this paper was submitted for the publication, the author received a letter by Y. Benyamini, where he announced similar results (not published yet).

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