Commentationes Mathematicae Universitatis Carolinae

Anthony W. Hager

# Real-valued functions on Alexandroff (zero-set) spaces

*Commentationes Mathematicae Universitatis Carolinae*, Vol. 16 (1975), No. 4, 755--769

Persistent URL: http://dml.cz/dmlcz/105664

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,4 (1975)

# REAL-VALUED FUNCTIONS ON ALEXANDROFF (ZERO-SET) SPACES[1]

Anthony W. HAGER[2], Middletown

Abstract: We give some approximation theorems describing the algebra of all real-valued "continuous" functions on a "space" in the sense of A.D. Alexandroff in terms of a generating subsystem. Corollaries include characterizations of such algebras (some known), and a concrete description of the functions on a subspace in terms of restrictions from the larger space. Topology generally lacks such theorems. The analogue of the Tietze-Urysohn extension theory is described, and the A-space analogues of topological pseudocompact, P-, and F-spaces are discussed briefly.

Key words: Alexandroff space, zero set space, cozeromorphism, inversion-closed, approximation, extension.

AMS: 54-00, 54C30, 54C45, 54C50, 54E15, 54H05

Ref. Ž.: 3.966

---

1. Alexandroff spaces. These spaces were introduced in [1] (under the name "completely normal spaces") and much of the basic theory was developed there.

1.1. Definition. A cozero-field on the set X is a family $\mathcal{A}$ of subsets of X satisfying

---

(1) This paper follows, to some extent, the recent paper [2] by R.L. Blair and the present author on somewhat similar topics in topology, and supercedes the unpublished manuscript [10].

(2) I am pleased to thank the Academies of Sciences of Czechoslovakia and the United States for support.

(a) $\emptyset , X \in \mathcal{A}$ .

(b) $\mathcal{A}$ is closed under finite intersection and countable union.

(c) If $A, B \in \mathcal{A}$ , with $(X - A) \cap (X - B) = \emptyset$ , then there are disjoint $A_1, B_1 \in \mathcal{A}$ with $A_1 \supset X - A$ and $B_1 \supset X - B$ .

(d) If $A \in \mathcal{A}$ , then there are $A_1, A_2, \ldots \in \mathcal{A}$ with $X - A = \bigcap_n A_n$ .

An Alexandroff space, or A-space, is a pair $\langle X, \mathcal{A} \rangle$ where $\mathcal{A}$ is a cozero-field on $X$ . The sets in $\mathcal{A}$ are called the cozero-sets of $\langle X, \mathcal{A} \rangle$ , and the complements are called zero-sets.

An A-map (or coz-map; or continuous function [1] ) $f: \langle X, \mathcal{A} \rangle \longrightarrow \langle Y, \mathcal{B} \rangle$ between A-spaces is a function with $f^{-1}(\mathcal{B}) \subset \mathcal{A}$ .

We shall see below why a cozero-field is so named.

1.1 (a) and (b) say that $\mathcal{A}$ is like a topology, but only closed under countable union. (c) is thus the analogue of normality, and (d) says that each "closed" set is a $G_\delta$. Consequently, any perfectly normal topological space "is" an A-space, and continuous maps between such spaces are A-maps.

Evidently, one gets a category with objects A-spaces and morphisms, the A-maps. A morphism set will generally be abbreviated to $A(X,Y)$ , and for $A(X,R)$ , we write just $A(X)$ (where $R$ is the real line).

In general, given $f: X \longrightarrow R$ , the cozero-set is $\operatorname{coz} f = \{x \mid f(x) \neq 0\}$ and the zero-set is $Zf = \{x \mid f(x) = 0\}$.

For $V \subset R^X$ , coz V = {coz f | f ∈ V} .

If X is a topological space, let aX be the A-space ⟨ X,coz C(X)⟩ , where C(X) is the set of real-valued continuous functions. This evidently defines a functor a . Note that for topological spaces X and Y , f ∈ C(X,Y) iff af ∈ A(aX,aY) when Y is Tychonoff, because coz C(Y) is a basis; thus the theory of "continuity" in A-spaces includes the theory of continuity in Tychonoff spaces. In the opposite direction, one may take the cozero-field of an A-space as the basis of a topology, thus defining a functor t . See [1].

Likewise, there is a functor similar to a , from uniform spaces to A-spaces, and more interestingly, at least two in the opposite direction: Given the A-space ⟨ X, $\mathcal{a}$ ⟩ , the finite and countable $\mathcal{a}$ -covers form respective bases for uniformities; these functors are full. Some of this is discussed in [11, 12, 7, 4, 5] .

We begin the examination of A(X) .

1.2. Proposition. If ⟨ X, $\mathcal{a}$ ⟩ is an A-space, A(X) is a uniformly closed inversion-closed vector lattice and ring, and BA(X) is a uniformly closed cbq vector lattice and ring.

The terminology: Let $V \subset R^X$ . BV is the subset of bounded functions. The uniform closure uc V consists of all limits of sequences from V which converge uniformly on X ; if uc V = V , V is uniformly closed. If f ∈ V and Zf = ∅ imply 1/f ∈ V , then V is inversion-closed. V is closed under bounded quotients (cbq) if f, g ∈ V ,

$Zg = \emptyset$ , and $f/g$ bounded imply $f/g \in V$ .

The proof of 1.2 is easy, and can be found in [1] or [13]. We shall see later that the conditions in 1.2 characterize the morphism sets A(X) and BA(X) .

The following guarantees that A(X) is large.

1.4. Proposition [1]. If $\mathcal{A}$ satisfies 1.1 (a),(b), (c), then whenever $A, B \in \mathcal{A}$ have $(X - A) \cap (X - B) = \emptyset$ , then there is $f: X \longrightarrow R$ with $f^{-1}(O) \in \mathcal{A}$ for each open O in R and $f(X - A) = 0$ , $f(X - B) = 1$ .

1.5. Corollary [1] . If $\langle X, \mathcal{A} \rangle$ is an A-space, then $\mathcal{A} = \mathrm{coz}\, A(X)$ .

1.4 is proved by the usual technique for Urysohn's Lemma. 1.5 follows by the argument used to show that a closed $G_\delta$ in a normal topological space is a zero-set.

2. Approximation and characterization. Some simple preliminaries are needed. The following will be used without explicit mention. We shall assume that all families of real-valued functions contain the constant function 1 .

2.1. Lemma. Let $V \subset R^X$ be a uniformly closed vector lattice. Then

(a) If $f \in V$ , then $|f| \in V$ .

(b) If $f \in V$ , then there is $g \in V$ with $0 \leq g \leq 1$ and $\mathrm{coz}\, g = \mathrm{coz}\, f$ .

(c) If $f \in V$ , then $f^{-1}(a,b) \in \mathrm{coz}\, V$ .

Proof. (a). $|f| = (f \vee 0) \vee ((-f) \vee 0)$ . (b). $g = |f| \wedge \wedge 1$ . (c). $f^{-1}(a,b) = \mathrm{coz}\, [(f - a) \vee 0] \wedge [(b - f) \vee 0]$.

2.2. Proposition. If $V \subset R^X$ is a uniformly closed vector lattice, then $\langle X, \mathrm{coz}\, V \rangle$ is an A-space.

Proof. $\emptyset = \mathrm{coz}\, 0 \in \mathrm{coz}\, V$ and $X = \mathrm{coz}\ 1 \in \mathrm{coz}\, V$.

The equation $\mathrm{coz}\, f_1 \cap \dots \cap \mathrm{coz}\, f_n = \mathrm{coz}\, (|f_1| \wedge \dots \wedge |f_n|)$ shows $\mathrm{coz}\, V$ is closed under finite intersection.

And $\bigcup_n \mathrm{coz}\, f_n = \mathrm{coz}\, (\sum_n |f_n| \wedge 2^{-n})$ (the series converging uniformly by the Weierstrass M-test) shows $\mathrm{coz}\, V$ is closed under countable union.

Let $f_1, f_2 \in V$ with $Zf_1 \cap Zf_2 = \emptyset$. Set $g_1 = (|f_2| - 2|f_1|) \vee 0$, $g_2 = (|f_1| - 2|f_2|) \vee 0$.

Then $\mathrm{coz}\ g_1 \supset Zf_1$, $\mathrm{coz}\, g_2 \supset Zf_2$, and $\mathrm{coz}\, g_1 \cap \mathrm{coz}\, g_2 = \emptyset$.

Finally, the equation $Zf = \bigcap_n \{x \mid |f|(x) < 1/n\}$ shows 1.1 (d).

Thus the question: what is $A(\langle X, \mathrm{coz}\, V \rangle)$ for $V$ as in 2.2 ?

2.3. Theorem. Let $V \subset R^X$ be a uniformly closed vector lattice. The following families coincide.

(a) $A(\langle X, \mathrm{coz}\, V \rangle)$.

(b) $\mathrm{uc}\, \{f/g \mid f, g \in BV,\ Zg = \emptyset\}$.

(c) The smallest uniformly closed vector lattice (and ring) $H(V)$ which is inversion-closed, and contains $V$.

Proof. To begin with, we show that the parenthetical condition in (c) follows from the rest of (c).

2.4. Lemma. (a) A uniformly closed vector lattice of bounded functions is a ring.

(b) A uniformly closed inversion-closed vector lattice is a ring.

<u>Proof</u>. Let $V$ be a vector lattice. To show that products from $V$ are in $V$, it suffices that $f \in V \Longrightarrow f^2 \in V$, by the equation $(f + g)^2 = f^2 + 2fg + g^2$.

(a). Let $V = BV$, and let $f \in V$. Let $\{h_n\}$ be a sequence of continuous piecewise linear functions on range $f$ which converges uniformly to the function $x \longmapsto x^2$. Then $\{h_n \circ f\}$ converges uniformly to $f^2$. Such $V$ is a vector lattice, each $h_n \circ f \in V$.

(b). Let $f \in V$. Then $Z(|f| \wedge 1) = \emptyset$, hence $1/(|f| + 1) \in BV$. By (a), $[1/(|f| + 1)]^2 \in BV$, and inverting again, $f^2 + 2|f| + 1 \in V$. Thus $f^2 \in V$.

Next, the smallest $H(V)$ in (c) exists: $R^X$ is such a family, $R^X \subset V$, and the intersection of such families is another.

We begin the proof proper. We abbreviate $A(\langle X, \text{coz } V \rangle)$ to $A$, and denote the object in (b) by $Q$.

$Q \subset H(V)$ : obvious.

$H(V) \subset A$ : By 2.1 (c) and the fact that each open set in $R$ is the union of a sequence of open intervals.

$A \subset Q$ : We show that if $f \in A$ and $\varepsilon > 0$, then there are $g, h \in BV$ with $|f(x) - g(x)/h(x)| < \varepsilon$ for each $x \in X$.

For each integer $i$, let $I_i$ be the open interval of length $\varepsilon/2$ with center $r_i = i(\varepsilon/4)$. Observe that $\{I_i\}_{-\infty}^{+\infty}$ is a cover of $R$ with the property that any $r \in I_i$ for at most two (consecutive) $i$'s; thus $f^{-1}(\{I_i\})$ is a cover of X with the same property in X . And each $f^{-1}(I_i) \in \text{coz } V$ .

For each i , choose $g_i \in V$ with $\text{coz } g_i = f^{-1}(I_i)$ and $0 \leq g_i \leq 1$ . Then $u = \sum g_i$ is well-defined (probably not in V ), and so are the functions $u_i = g_i/u$ . Then

$$|\sum r_i u_i(x) - f(x)| < \varepsilon \quad \text{for each } x \in X ,$$

because: $\sum u_i = 1$ , so $\sum r_i u_i - f = \sum (r_i - f) u_i$ . Given x, x belongs to at most two consecutive coz $u_i$ ( = coz $g_i$ ), and $\sum (r_i - f(x)) u_i(x)$ has at most two non-zero terms, each of absolute value $< \varepsilon/2$ .

Let $\alpha_i = [2^i (1 \vee (|r_{i-1}| + |r_i| + |r_{i+1}|))]^{-1}$ , and let $w = \sum \alpha_i g_i$ . Then $\sum r_i u_i = w \sum r_i g_i / w \sum g_i$ . We show that $g = w \sum r_i g_i$ and $h = w \sum g_i$ are in BV .

Consider a more general product of the form of these, $(\sum_i \beta_i g_i)(\sum_j \gamma_j g_j) = \sum_i \beta_i (\sum_j \gamma_j g_j) g_i$ . Since $g_j g_i \neq 0$ for at most $j = i - 1, i, i\ 1$ , this becomes $\sum_i \beta_i (\gamma_{i-1} g_{i-1} + \gamma_i g_i + \gamma_{i+1} g_{i+1}) g_i$ , which we call $\sum_i w_i$ . By 2.4 (a) , each $w_i \in BV$ . We show that the series converges uniformly for coefficients $\{\beta_i\}, \{\gamma_j\}$ chosen so as to produce g and h . Since

$$|w_i| \leq |\beta_i| (|\gamma_{i-1}| + |\gamma_i| + |\gamma_{i+1}|) :$$

For g , we choose $\beta_i = \alpha_i$ and $\gamma_j = r_j$ ; then $|w_i| \leq 2^{-i}$ . For h , we choose $\beta_i = \alpha_i$ , $\gamma_j = r_j$ ; then $|w_i| \leq 2^{-i}$ .

The proof is complete.

2.3 has evolved from restricted versions or variants in [8] and [2]. Related, and partially overlapping results appear in § 41 of [13], and in [15]; these proofs do not bear much resemblance to that above.

We mention some other constructions of $A(\langle X, \mathrm{coz}\ V \rangle)$ from $V$ ( $V$ being a uniformly closed vector lattice). We shall only sketch the proofs.

Let $\overline{V}$ (respectively, $\underline{V}$ ) denote the collection of all limits of pointwise convergent increasing (respectively, decreasing) sequences from $V$ .

2.5. Theorem. $BA(\langle X, \mathrm{coz}\ V \rangle) = B(\overline{V} \cap \underline{V})$ .

(Note that to construct an $A(X)$ it suffices to construct $BA(X)$ , because $A(X) = \{f/g \mid f,g \in BA(X),\ Zg = \emptyset\}$.)

Proof. In [9], it is shown that if $f \in A(\langle X, \mathrm{coz}\ V \rangle)$ and $f$ is bounded below, then $f \in \overline{V}$ . This and its "dual" give the inclusion " $\subset$ " in 2.5. The reverse inclusion follows from the elementary fact that if $f \in V$ then $\{x \mid f(x) > r\} \in \mathrm{coz}\ V$ for each $r \in R$ , and "dually".

Note the use of "lower semi-cozero functions" here. More explicitly, Mauldin [14] has shown that for $f$ bounded below, $\{x \mid f(x) > r\} \in \mathrm{coz}\ V$ for each $r \in R$ iff $f \in \overline{V}$ . Of course, this can be used to prove 2.5.

The next theorem uses Frolík's "strong continuous convergence" (which we indicate by " $f_n \xrightarrow{sc} f$" ). See [4] for the definition.

2.6. Theorem. These conditions on $f$ are equivalent:

(a) $f \in A(\langle X, \mathrm{coz}\ V \rangle)$.

(b) $f = g \circ (f_n)$ , for some sequence $\{f_n\} \subset V$ and

$g \in A((f_n)(X))$ .

(c) There is a sequence $\{f_n\} \subset V$ with $f_n \xrightarrow{sc} f$ .

Here, $(f_n)$ denotes the reduced product, or diagonal map, of $X$ into $R^{\aleph_0}$ , and $(f_n)(X)$ is the range. This range is metric, so that $A((f_n)(X)) = C((f_n)(X))$ .

Proof. (a) $\Longrightarrow$ (c). Write $f = f^+ - f^-$ , and by the device from [9] used in the proof of 2.6, choose $\{g_n\}$ , $\{h_n\} \subset V$ with $g_n \uparrow f^+$ and $h_n \downarrow f^-$ . Then $g_n - h_n \xrightarrow{sc} f$ . (c) $\Longrightarrow$ (b). Let $f_n \xrightarrow{sc} f$ , and define $g: (f_n)(X) \longrightarrow R$ by $g((f_n)(x)) = f(x)$ . Using sequences in $(f_n)(X)$ , continuity of $g$ is easily verified. (b) $\Longrightarrow$ (a). One checks easily that $(f_n)$ is an A-map, and hence so is $g \circ (f_n)$ . (That $(f_n)$ is an A-map uses separability of the range. In general, the reduced product of even two A-maps need not be an A-map. See [12].)

The equivalence of (a) and (c) in 2.6 is the "constructive version" of a characterization in [4]. [4] includes some other closely related ideas.

Each of the foregoing constructions yields immediately a characterization of the morphism sets $A(X)$ :

2.7. Corollary. Let $V \subset R^X$ . The following are equivalent:

(a) $V = A(\langle X, \mathcal{a} \rangle)$ for some cozero-field $\mathcal{a}$ on $X$ .

(b) $V = A(\langle X, \operatorname{coz} V \rangle)$ .

(c) $V$ is a uniformly closed inversion-closed vector lattice (or ring).

(d) $V$ is "sc-closed".

(e) $V$ is "composition-closed".

Likewise, the morphism sets $BA(X)$ can be characterized, notably as the uniformly closed cbq vector lattices $V$ with $V = BV$, or as those $V$ with $V = B(\overline{V} \cap \underline{V})$.

3. Functions on subspaces. The main observation here (a simple corollary of 2.3) is that for $X$ an A-space and $S$ an A-subspace (defined shortly) the functions in $A(S)$ have an explicit description in terms of the restrictions of functions in $A(X)$. This should be compared with topology, where for $S \subset X$, $C(S)$ generally bears no concrete relation to $C(X)$. The present simplification results directly from the equality $\operatorname{coz} A(S) = \operatorname{coz} A(X) | S$, the analogue of which fails in topology.

Notation: For $\langle X, \mathcal{A} \rangle$ an A-space and $S \subset X$, $\mathcal{A} | S = \{A \cap S \mid A \in \mathcal{A}\}$, and for $\langle Y, \mathcal{B} \rangle$ another A-space, $A(X,Y) | S$ is the set of restrictions $f | S$, for $f \in A(X,Y)$.

3.1. Proposition. If $\langle X, \mathcal{A} \rangle$ is an A-space, and $S \subset X$, then $\mathcal{A} | S$ is a cozero-field on $S$. So $\langle S, \mathcal{A} | S \rangle$ is an A-space.

Proof. It is obvious that $\mathcal{A} | S$ satisfies Conditions 1.1 (a),(b),(d). (c) is more difficult, but proved exactly as one proves that a perfectly normal topological space is hereditarily normal; see [3] for a sketch of this.

So, $\langle S, \mathcal{A} | S \rangle$ is said to be an A-subspace of $\langle X, \mathcal{A} \rangle$. We shall write $S \subset X$ when no ambiguity seems likely.

3.2. Corollary. If $S \subset X$, then

(a) for any $X$, $A(X,Y) | S \subset A(S)$ ;

(b) $\operatorname{coz} A(S) = \operatorname{coz} (A(X) | S) = (\operatorname{coz} A(X)) | S$.

*Proof.* (a) is obvious. For (b): If $\mathcal{a}$ is the cozero-field of $X$, then $\mathcal{a}|S$ is the cozero-field of $S$, and thus $\mathcal{a}|S = \operatorname{coz} A(S)$ by 1.5.
Since $\mathcal{a}|S = \operatorname{coz}(A(X)|S) = (\operatorname{coz} A(X))|S$, (b) follows.

3.3, *Lemma.* Let $S \subset X$, let $f_1, f_2, \ldots \in A(X)|S$, and let $f\colon S \to R$ be a function. If $f_n \to f$ uniformly on $S$, then $f \in A(X)|S$.

The usual proof for continuous functions works here; see [2].

3.4. *Corollary.* Let $S \subset X$. Then $A(X)|S$ is a uniformly closed vector lattice (with $\operatorname{coz} A(X)|S = \operatorname{coz} A(S)$).

*Proof.* Obviously, $A(X)|S$ is a vector lattice. That it is uniformly closed follows from 3.3.

3.5. *Theorem.* Let $S \subset X$. If $f \in A(S)$ and $\varepsilon > 0$, then there are $g, h \in A(X)$ with $Z(h) \cap S = \emptyset$ and

$$|f(\ ) = g(\ )/h(\ )| < \varepsilon \quad \text{for each } s \in S .$$

That is, $A(S) = \operatorname{uc}\{g/h \mid g, h \in A(X)|S \text{ and } Z(h) = \emptyset\}$.

*Proof.* By 3.4 and 2.3.

Similar theorems can be derived from 2.5 and 2.6.

4. *Extension theorems.* We now describe an extension theory for A-spaces analogous to that for topology originating with Tietze and Urysohn. The development follows [2].

4.1. *Theorem.* Let $S \subset X$. Then $A(S) = A(X)|S$ (respectively, $BA(S) = BA(X)|S$) iff $A(X)|S$ is inversion-closed (respectively, cbq).

*Proof.* Immediate from 3.5 and 3.3.

This uses the approximation theorem 2.3.

A more usual argument yields a more usual theorem, 4.3 below.

4.2. __Lemma__. Let $E_1, E_2 \subset X$. The following are equivalent.

(a) There are disjoint zero-sets $Z_1, Z_2$ of $X$ with $E_1 \subset \subset Z_1$ and $E_2 \subset Z_2$.

(b) There is $f \in A(X)$ (with $0 \leq f \leq 1$) with $f(E_1) = 0$ and $f(E_2) = 1$.

The usual proof for topology works here; see [6].

As in topology (e.g. [6]) subsets $E_1$ and $E_2$ which satisfy the conditions 4.2 are said to be completely separated in $X$.

4.3. __Theorem__. Let $S \subset X$.

A. $BA(X) | S = BA(S)$ iff disjoint zero-sets of $S$ are completely separated in $X$.

B. $A(X) | S = A(S)$ iff $S$ is completely separated from each disjoint zero-set.

__Proof__. A. can be proved by the usual Urysohn technique described in [6], or by the somewhat different method in 3.4 of [2].

To prove B., first note that the separation hypothesis in B. implies that in A ; the proof then proceeds as in 3.4 of [2]. Alternatively, a direct proof of B. from 2.3 is possible; see page 47 of [2].

The results for topology described in [2] which correspond to 4.1 and 4.3 can be derived as follows: For $X$ a topological space, $\langle X, \text{coz } C(X) \rangle$ is an A-space with $A(X) = C(X)$. But a topological subspace $S$ need not be an A-subspace; the condition that it be is called "z-embedded" in [2]. So for example: $C(X) | S = C(S)$ iff $S$ is completely separated from each disjoint zero-set and $S$ is z-embedded; this is part of 3.6 of [2], and is immediate from 4.3 B. The reader can easily finish the comparison with [2].

5. <u>On special cases</u>. We conclude with some discussion of A-spaces which arise from consideration of the conditions in 4.1 and 4.3. Again, the discussion is modeled on [2] (§ 4 ), and so we shall omit proofs.

The Alexandroff compactification $\beta X$ of the A-space $X$ is the space of zero-set ultrafilters of $X$. It has the properties: $\beta X$ is a compact A-space; $X$ is a dense A-subspace; each A-map of $X$ to a compact A-space has a unique A-extension over $\beta X$. See [1]. (Thus $\beta X$ is the compact reflection in the category of A-spaces.)

5.1. <u>Proposition</u>. The following conditions on the A-space $S$ are equivalent.

(a) $S$ is pseudocompact: $A(S) = BA(S)$.

(b) The cozero-field of $S$ is semi-compact: each countable cozero cover has a finite subcover.

(c) Each zero-set of $\beta S$ meets $S$.

(d) Whenever $S$ is an A-subspace of $X$, then

$A(X) | S = A(S)$ (or, $BA(X) | S = BA(S)$ ).

See 4.3 of [2], and Gordon's nice theorem [7] that a pseudocompact A-space has only one compactification. Other equivalent conditions are given in [12].

5.2. Proposition. The following condition on the A-space $X$ are equivalent.

(a) For each $S \subset X$ (or, for each cozero set $S \subset X$ ), $A(X) \mid S = A(S)$ .

(b) The cozero-field of $X$ is a $\sigma$-field.

And then $\beta X$ is the Stone space of coz $A(X)$ , hence basically disconnected.

5.3. Proposition. The following conditions on the A-space $X$ are equivalent.

(a) For each $S \subset X$ (or, for each cozero set $S \subset X$) , $BA(X) \mid S = BA(S)$ .

(b) $t\,\beta X$ is an F-space.

These describe the A-space analogous of topological P-spaces and F-spaces. See 4.5 of [2]. Virtually all the other equivalent conditions from topology carry over; see Chapter 14 of [6].

R e f e r e n c e s

[1] A.D. ALEXANDROFF: Additive set functions in abstract spaces, Mat.Sbornik 50(1940), 30-348; 51(1941), 563-628; 55(1943),169-238.

[2] R.L. BLAIR and A.W. HAGER: Extensions of zero-sets and of real-valued functions, Math. Zeit. 136(1974), 41-52.

[3] R. ENGELKING: Outline of General Topology, Amsterdam, 1968.

[4] Z. FROLÍK: Three uniform spaces associated with real-valued functions, Proc. Rome conference on rings of continuous functions 1973, to appear.

[5] Z. FROLÍK: On uniform spaces, Comment. Math. Univ. Carolinae 16(1975),189-199.

[6] L. GILLMAN and M. JERISON: Rings of Continuous Functions, D. van Nostrand Co., 1960.

[7] H. GORDON: Rings of functions determined by zero-sets, Pac. J. Math. 36(1971), 133-157.

[8] A.W. HAGER: On inverse-closed subalgebras of C(X), Proc. London Math. Soc. (3)19(1969), 233-257.

[9] A.W. HAGER: An approximation technique for real-valued functions, Gen. Top. and Appl. 1(1971),415-418.

[10] A.W. HAGER: An approximation technique for real-valued functions, 2, preprint 1972.

[11] A.W. HAGER: Some nearly fine uniform spaces, Proc. London Math. Soc. (3)28(1974), 517-546.

[12] A.W. HAGER: Uniformities induced by proximity, cozero- and Baire sets, to appear.

[13] F. HAUSDORFF: Set Theory (Chelsea Reprint) New York, 1957.

[14] R.D. MAULDIN: On the Baire system generated by a linear lattice of functions, Fund. Math. 68(1970),51-59.

[15] S.G. MRÓWKA: Characterization of classes of functions by Lebesgue sets, Czech. Math. J. 19(1969), 738-744.

Wesleyan University
Middletown
Connecticut
U.S.A.

and

Mathematics Institute
of the Czechoslovak
Academy of Sciences
Prague, Czechoslovakia

(Oblatum 26.3. 1975)