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Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 4, 735--743

Persistent URL: http://dml.cz/dmlcz/105662

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,4 (1975)

ON RICH MONOIDS

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Abstract: The hereditarity of the poorness of monoids is studied in the first part of the paper. Every rich monoid is a submonoid of some poor monoid; moreover, every finitely generated free monoid is a submonoid of some poor monoid with only two generators. The remaining part demonstrates a sort of unreducibility of the whole problem of rich monoids to the monoids with two generators.

Key words: Category, functor, monoid, unary algebra.

AMS: 18B15 Ref. Ž.: 2.726.3

Introduction. To describe the contents of the paper, let us first recall some notions. A category C is called algebraic if there exists a full embedding of C into some category of algebras and all their homomorphisms. A category is said to be binding ([1]) if every algebraic category can be embedded into it. A small category c is said to be rich ([2],[3]) if the functor category Set^c is binding, otherwise it is called poor.

The complete characterization of rich thin categories (i.e. preorders) has been given in [2], its counterpart yet not being known for another important case of one-object categories - monoids. So far only special classes of rich monoids have been described, e.g., in [4] rich monoids with two idempotent generators φ, ψ are characterized: Such a monoid is rich if and only if it has as a factormonoid one of the monoids M_k defined by the identities $\varphi = \varphi^2 = (\varphi\psi)^k\varphi$, $\psi = \psi^2 = (\psi\varphi)^k\psi$ with $k \geq 3$.

For an arbitrary monoid $M = \Sigma^*/Q$ defined by the set Σ of its generators and the set Q of identities in the alphabet Σ we can consider the functor category Set^M as a category of algebras with the set Σ of unary operations fulfilling the identities from Q . Thus, the problem of rich monoids is just the question which monoids of unary operations are large (or, better, intricate) enough for the corresponding categories of algebras to be sufficiently comprehensive, i.e. to contain any algebraic category.

As to cardinality, every rich monoid has at least five elements [5], and every set of its generators has at least two elements. Since a large majority of the results on rich monoids obtained so far concerns the monoids with two generators, this could make the impression that the whole question of the richness of monoids might be reducible, in a sense, to the range of monoids with two generators. In the second paragraph of this paper we present an example demonstrating that this is not the case.

Since the factorization of monoids implies full embedding of the corresponding categories in the converse direction, the factormonoid of a poor monoid is poor. The first paragraph of the present paper is concerned with the behaviour of the richness of monoids as to their inclusions. While the commutative monoid with two generators is an example of the hereditarily poor monoid, we shall show that generally the poorness of monoids is not hereditary, and that even rich monoid is a submonoid of some poor monoid. Moreover, we shall show that every finitely generated free monoid can be embedded into some poor monoid with only two generators. The related questions concerning the "inheriting" of richness of monoids from their factormonoids are studied in [6].

I want to express my gratitude to doc. Věra Trnková for her encouragement in my work.

1. The hereditarity of the poorness of monoids.

In view of the motivation based on unary operations, every monoid is supposed to be given by some set Σ of its generators and some set Q of identities in the alphabet Σ ; then it is denoted by Σ^*/Q . Let $M = \Sigma^*/Q$, $w \in \Sigma^* = \Sigma^*/\emptyset$. Then $[w]_M$ denotes the element of M in the usual sense; brackets and index M a re sometimes omitted.

An object of a category C is called rigid if its only endomorphism in C is the identity. From [7] and [8] it follows that any binding category contains a proper class of mutually non-isomorphic rigid objects.

1.1. Proposition. Every monoid M can be embedded into some poor monoid.

Proof: Let $M = \Sigma^*/Q$, denote by R the set of all identities of the form $\sigma\alpha = \alpha\sigma = \alpha$ for all $\sigma \in \Sigma$, denote $M_1 = \Sigma \cup \{\alpha\} / Q \cup R$. Then 1.1 follows from 1.2 and 1.3 .

1.2. Lemma. The formula $f([\sigma]_M) = [\sigma]_M$ defines a homomorphism $f: M \longrightarrow M_1$, which is 1-1 .

Proof is obvious.

1.3. Lemma. M_1 is poor.

Proof: Let $A = (X, \Sigma \cup \{\alpha\}) \in \mathrm{Set}^{M_1}$ be a rigid algebra. Then α is its endomorphism, hence $\alpha = \mathrm{id}$. Consequently, $\sigma = \mathrm{id}$ for all $\sigma \in \Sigma$. Thus every constant mapping of X into itself is an endomorphism of A , which implies $\mathrm{card}\, X = 1$, in other words, the category Set^{M_1} has only trivial rigid objects.

1.4. Proposition. Every finitely generated free monoid can be embedded into a poor monoid with two generators.

Proof: It is well known that every finitely generated free monoid can be embedded into the free monoid with two generators, so 1.4 follows from the next two lemmas.

1.5. Lemma. Let $M_2 = \{\sigma, \tau\}^*$, $M_3 = \{\alpha, \beta\}^* / \{\alpha\beta\alpha = \alpha^2\beta\alpha = \beta\alpha\beta\alpha\}$. Then the formulas $g(\sigma) = [\alpha]$, $g(\tau) = [\beta^2]$ define a homomorphism $g: M_2 \longrightarrow M_3$, which is 1 - 1 .

Proof: The elements of $g(M_2) \subset M_3$ are represented only by such words in the alphabet $\{\alpha, \beta\}$ which do not contain the subword $\alpha\beta\alpha$. But the identities defining M_3 cannot be applied to such words, hence, g is 1-1 .

1.6. Lemma. M_3 is poor .

Proof: Let $A = (X, \alpha, \beta) \in \mathrm{Set}^{M_3}$ be a rigid algebra, $x \in X$, $y = \alpha\beta\alpha(x)$. Then $\alpha(y) = \beta(y) = y$, hence the constant mapping of X onto y is an endomorphism of A , so $X = \{y\}$. Again, Set^{M_3} has only trivial rigid algebras.

1.7. Problem. Can every finitely generated (non-free) monoid be embedded into a poor monoid with two generators?

2. The rich monoid with three generators, whose each pair of elements generates a poor submonoid.

Let $M_4 = \{\alpha, \beta, \gamma\}^*/Q_4$, where

$$Q_4 = \left\{\begin{array}{llllll} \alpha = \alpha^2 & = \alpha\beta\alpha & = \alpha\gamma\alpha & = \alpha\beta\gamma\alpha & = \alpha\gamma\beta\alpha \\ \beta = \beta^2 & = \beta\alpha\beta & = \beta\gamma\beta & = \beta\alpha\gamma\beta & = \beta\gamma\alpha\beta \\ \gamma = \gamma^2 & = \gamma\alpha\gamma & = \gamma\beta\gamma & = \gamma\alpha\beta\gamma & = \gamma\beta\alpha\gamma \end{array}\right\}.$$

2.1. Theorem. The monoid M_4 is a rich monoid with three generators, whose each submonoid with two generators as well as each of its factormonoids with two generators are poor.

Proof: is given in the following lemmas.

2.2. Lemma. The monoid $M_5 = \{\mu, \nu\}^*/\{\mu = \mu^2 = \mu\nu\mu,\ \nu = \nu^2 = \nu\mu\nu\}$ is poor.

Proof: We show that any rigid algebra $A = (X, \mu, \nu) \in \mathrm{Set}^{M_5}$ has at most two elements. Suppose $X \neq \emptyset$. Define $K = \{x \in X;\ \mu(x) = x\}$, $L = \{x \in X;\ \nu(x) = x\}$. If

$K \cap L \neq \emptyset$, then the constant mapping of X on some $x \in K \cap \cap L$ is an endomorphism of A , hence $X = \{x\}$, i.e. card $X = 1$. If $K \cap L = \emptyset$, choose $x \in X$ and denote $z = \mu(x)$, $y = \nu(z)$. Then $\mu(z) = z = \mu(y)$, $\nu(z) = y = \nu(y)$. Now one can verify that the mapping $f: X \longrightarrow X$ defined by

$f(t) = z$ whenever $t \in K$,

$f(t) = y$ otherwise

is an endomorphism of A . Hence $X = \{z, y\}$, i.e. card $X = 2$.

2.3. Lemma. If $s, s' \in M_4$, then $s = s^2 = ss's$.

Proof: Since M_4 is symmetric with respect to α , β , γ , it is sufficient to verify the above equation only for $s \in \{1, \alpha, \alpha\beta, \alpha\beta\gamma\}$, $s' \in M_4 = \{1, \alpha, \beta, \gamma, \alpha\beta, \alpha\gamma, \beta\alpha, \beta\gamma, \gamma\alpha, \gamma\beta, \alpha\beta\gamma, \alpha\gamma\beta, \beta\alpha\gamma, \beta\gamma\alpha, \gamma\alpha\beta, \gamma\beta\alpha\}$. The computation is rather long, but very easy and therefore left to the reader.

2.4. Lemma. i) Each submonoid with two generators of M_4 is poor. ii) Each factormonoid with two generators of M_4 is poor.

Proof: i) By the previous lemma it is a factormonoid of the poor monoid M_5 .

ii) Let $h: M_4 \longrightarrow M$ be an epimorphism, M having two generators m, n . Choose $p, q \in M_4$ so that $h(p) = m$, $h(q) = n$. Then M is a factormonoid of the submonoid M' of M_4 generated by p, q . Since M' is poor, M is also poor.

2.5. Lemma. M_4 is rich.

Proof: Let Graph denote the category of all directed graphs and their compatible mappings. According to [6], it is sufficient to construct a full embedding Φ : : Graph $\longrightarrow$ Set^{M_4}. Define

$$\Phi(X,R) = (Y,\alpha,\beta,\gamma) \ , \ Y = (R \times \{1,2,3\}) \cup (X \times \{4,5\}),$$

denote by Π_1, $\Pi_2 : R \longrightarrow X$ the projections. If $r \in R$, $x \in X$, then we put

$$\alpha(r,1) = (\Pi_1(r),4), \ \beta(r,1) = (r,2), \ \gamma(r,1) = (\Pi_2(r),4),$$

$$\alpha(r,2) = \beta(r,2) = (r,2), \ \gamma(r,2) = (r,3) \ ,$$

$$\alpha(r,3) = \beta(r,3) = (r,2), \ \gamma(r,3) = (r,3) \ ,$$

$$\alpha(x,4) = \gamma(x,4) = (x,4), \ \beta(x,4) = (x,5) \ ,$$

$$\alpha(x,5) = \gamma(x,5) = (x,4), \ \beta(x,5) = (x,5) \ .$$

One can verify that $A = (Y,\alpha,\beta,\gamma)$ is really an algebra from Set^{M_4}. Let $f: (X,R) \longrightarrow (X',R')$ be a compatible mapping. Define $\Phi(f) = g$, where $g(x,y,i) = (f(x), f(y),i)$ whenever $(x,y) \in R$, $i \in \{1,2,3\}$, $g(x,i) = (f(x),i)$ whenever $x \in X$, $i \in \{4,5\}$. Clearly, Φ is an embedding. We have to prove that Φ is full. Let $g: \Phi(X,R) \longrightarrow \Phi(X',R')$ be a homomorphism. Denote $\Phi(X',R') = (Y',\alpha',\beta',\gamma')$. Since $X' \times \{4\}$ is just the set of all $z \in Y'$ such that $\alpha'(z) = \gamma'(z) = z$, we have $g(X \times \{4\}) \subset X' \times \{4\}$. Define a mapping $f: X \longrightarrow X'$ by $g(x,4) = (f(x),4)$. Then $g(x,5) = g\beta(x,4) = \beta' g(x,4) = \beta'(f(x),4) = (f(x),5)$. Since $R' \times \{2\}$ is just the set of all $z \in Y'$ such that

$\alpha'(z) = \beta'(z) = z$, we have $g(R \times \{2\}) \subset R' \times \{2\}$. Since $R' \times \{1\}$ is just the set of all $z \in Y'$ such that $\alpha'(z) \in X' \times \{4\}$, $\beta'(z) \in R' \times \{2\}$, we have $g(R \times \{1\}) \subset R' \times \{1\}$. Now, it is easy to show $g = \Phi(f)$.

The proof of the theorem is concluded.

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(Oblatum 30.5. 1975)

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