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LINEAR AND NONLINEAR VARIATIONAL INEQUALITIES ON HALF-SPACES

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Abstract: There are given necessary and sufficient conditions upon the right hand sides to be the linear variational inequality solvable on a given half-space. The nonlinear inequalities are also investigated. The abstract results are applied to the inequalities involving the ordinary differential operators.

Key-words: Variational inequalities, Schauder fixed point theorem, weak solutions of variational inequalities with differential operators.

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1. Introduction. Let K be a closed convex non-empty subset of a Hilbert space H. Let S be an operator (generally nonlinear) acting from K into H. The well-known theorem (see e.g. [1], Chapter II, Section 8) says that under some "continuity conditions on S" (e.g. S is so-called pseudomonotone) and if S is coercive (i.e. there exists $v_0 \in K$ such that

$$(1.1) \qquad \lim_{\substack{\|u\| \to \infty \\ u \in K}} \frac{(Su, u - v_0)}{\|u\|} = +\infty \quad)$$

then the variational inequality

(1.2) $(Su - f, v - u) \geq 0$ for all $v \in K$

possesses a solution $u \in K$ for arbitrary $f \in H$.

It is easy to see that the condition (1.1) is not necessary for the validity of the previous assertion.

The purpose of this note is following. We consider that K is a half-space in H and S is linear. We define a set $\mathrm{Ker}_K S$ and if we denote by $R_K(S)$ the set of all $f \in H$ for which the variational inequality (1.2) has a solution then we prove

" $\mathrm{Ker}_K S = \{0\}$ if and only if $R_K(S) = H$ ".

If $\mathrm{Ker}_K S \neq \{0\}$ then the description of $R_K(S)$ is given. So we obtain the analogous assertions as the Fredholm theorems for linear equations.

Using the results of the type above, by applying the Schauder fixed point theorem, we obtain some results about the solvability of the nonlinear variational inequality (1.2), where S is the sum of a linear operator and a nonlinear compact perturbation.

Finally, we apply the abstract results to the variational inequalities involving the ordinary differential operators of the second order.

The paper is an excursion to the problems which are not solved up to now by the authors' best knowledge. So we also formulate some open problems which are in the connection with our problems and the solving of which seems to be interesting and useful.

2. Notation, terminology. Let H be a non-trivial real Hilbert space with the inner product $(.,.)$ and the norm $\|u\| = (u,u)^{1/2}$. Unles stated otherwise, we denote a bounded linear self-adjoint operator from H into H by A. Throughout the paper we fix $\xi \in H$, $\|\xi\| = 1$, and denote

$N = \{u \in H\ ;\ (u, \xi) = 0\}$,

$K = \{u \in H\ ;\ (u, \xi) \geq 0\}$,

$K^{0} = \{u \in H\ ;\ (u, \xi) > 0\}$.

Let $R_K(A)$ be a set of all $f \in H$ for which there exists at least one $u \in K$ satisfying the inequality

$(Au - f, v - u) \geq 0$ for all $v \in K$.

Finally, put

$Ker_K A = \{u \in H\ ;\ (Au, v - u) \geq 0$ for all $v \in K\}$.

3. Auxiliary lemmas. Before formulating and proving the results mentioned in Section 1 we prove some simple lemmas.

3.1. Lemma. $f \in R_K(A)$ if and only if there exist $n_0 \in N$, $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha\beta = 0$ and

$$f = A(n_0 + \alpha \xi) - \beta \xi .$$

Proof. Let $f \in R_K(A)$. Then

(3.1) $\qquad (Au - f, v - u) \geq 0$ for all $v \in K$,

where $u = n_0 + \alpha\xi$ for some $n_0 \in N$ and $\alpha \geq 0$. Putting $v = u + t\xi$, $t \geq 0$, we obtain from (3.1)

(3.2) $\qquad (Au - f, \xi) \geq 0$.

Putting $v = u + n$, $n \in N$ is arbitrary, we get

$(Au - f, n) \geq 0$. Similarly $(Au - f, -n) \geq 0$ and thus

(3.3) $\qquad (Au - f, n) = 0$ for all $n \in N$.

Therefore $Au - f = \beta\xi$ for some real β and the inequality (3.2) implies $\beta \geq 0$. Putting $v = 0$ and $v = 2u$ we obtain from (3.1)

(3.4) $\qquad (Au - f, u) = 0$

and so $0 = (\beta\xi, n + \alpha\xi) = \alpha\beta$.

Conversely, let there exist $n_0 \in N$, $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha\beta = 0$ and $f = A(n_0 + \alpha\xi) - \beta\xi$. Then for $u = n_0 + \alpha\xi \in K$ we have

$(Au - f, v - u) = \beta(\xi, v - n_0 - \alpha\xi) = \beta(\xi, v) \geq 0$

for all $v \in K$.

The following lemma explains the notion of $\mathrm{Ker}_K A$.

3.2. Lemma. $\mathrm{Ker}_K A = \mathrm{Ker}\, A \cup \bigcup_{\alpha > 0} \alpha\Gamma$,

where

$\Gamma = \{\omega \in H ; (\omega, \xi) \leq 0$ and $A\omega = \xi\}$.

Proof. Put $\mathcal{M} = \mathrm{Ker}\, A \cup \bigcup_{\alpha > 0} \alpha\Gamma$.

Let $u \in \mathrm{Ker}\, A$. Then obviously $u \in \mathrm{Ker}_K A$. Let $Au = \alpha\xi$, where $\alpha > 0$, $(u, \xi) \leq 0$. Then

$(Au, v - u) = \alpha[(\xi, v) - (\xi, u)] \geq 0$

for all $v \in K$, i.e. $u \in \mathrm{Ker}_K A$. Thus $\mathcal{M} \subset \mathrm{Ker}_K A$. Conversely, let $u \in \mathrm{Ker}_K A$. If $u \in K^0$ then, by Lemma 3.1, $Au = 0$, i.e. $u \in \mathrm{Ker}\, A$ and thus $u \in \mathcal{M}$. If $u \in N$ then $(Au, n) = 0$ (putting $v = u + n$) for arbitrary $n \in N$ and $(Au, \xi) \geq 0$ (putting $v = u + \xi$) . Thus $Au = \alpha\xi$, $\alpha \geq 0$, and $u \in \mathcal{M}$. If $-u \in K^0$ then there exist $n_0 \in N$, $\alpha > 0$ such that $u = n_0 - \alpha\xi$. Hence

(3.5) $$(Au, \xi) \geq 0$$

(putting $v = n_0$) and

(3.6) $$(Au, \alpha\xi) \geq t(Au, n)$$

(putting $v = -tn + n_0$) for arbitrary real t and all $n \in N$. It follows that

(3.7) $$(Au, n) = 0$$

for all $n \in N$. The relation (3.7) implies $Au = \gamma\xi$ which together with (3.5) gives $\gamma \geq 0$ what is nothing else than $u \in \mathfrak{M}$.

4. Linear inequalities.

4.1. Theorem. Let A be a self-adjoint operator on H. Then $R_K(A) = H$ if and only if $\mathrm{Ker}_K A = \{0\}$. Moreover, if $\mathrm{Ker}_K A = \{0\}$ then for arbitrary $f \in H$ there exists the uniquely determined $Bf \in K$ such that

(4.1) $$(ABf - f, v - Bf) \geq 0 \quad \text{for all} \quad v \in K.$$

The mapping B from H into K is continuous, nonlinear and

(4.2) $$\|Bf\| \leq \|A^{-1}\| \cdot \|f\| + \frac{\|f\| \cdot \|A^{-1}\xi\|^2}{(\xi, A^{-1}\xi)}$$

for arbitrary $f \in H$ (A^{-1} denotes the inverse of A, $\|A^{-1}\|$ its norm).

Proof. First, from Lemma 3.1 it follows that $R_K(A) = H$ if and only if the following three conditions are fulfilled:

(4.3) $$\operatorname{codim} A(N) = 1 ,$$

(4.4) $$A\xi \notin A(N) ,$$

(4.5) $$\xi \in A(K^{o}) .$$

It is easy to see that the conditions (4.3) and (4.4) are equivalent to

(4.6) $$\operatorname{Ker} A = \{0\} .$$

Let now $\operatorname{Ker}_K A = \{0\}$. Then (4.6) and (4.5) are fulfilled and therefore $R_K(A) = H$. If $R_K(A) = H$ then (4.5) implies that any solution of $Au = \alpha\xi$ with $\alpha > 0$ lies in K^{o} and thus together with (4.6) we have $\operatorname{Ker}_K A = \{0\}$.

To prove the second part of the theorem suppose that $\operatorname{Ker}_K A = \{0\}$, $f \in H$, and let $A^{-1}\xi = \omega$. According to Lemma 3.1 we have

$$n_0 + \alpha\xi - \beta\omega = A^{-1}f ,$$

where

$$\alpha = (f,\omega) , \ \beta = 0 \ \text{ if } \ (f,\omega) > 0 ,$$

$$\alpha = 0 , \ \beta = \frac{(f,\omega)}{(\omega,\xi)} \ \text{ if } \ (f,\omega) < 0 ,$$

$$\alpha = 0 , \ \beta = 0 \ \text{ if } \ (f,\omega) = 0 .$$

From this it follows the existence of a mapping $B: f \mapsto n_0 + \alpha\xi$ with the property (4.1) and also the estimate (4.2). The continuity of B is obvious.

Before the description of $R_K(A)$ in the case of

$\mathrm{Ker}_K A \neq \{0\}$ we mention that if A is a self-adjoint operator on H then

$$A(H) = (\mathrm{Ker}\ A)^{\perp} \quad ,$$

where

$(\mathrm{Ker}\ A)^{\perp} = \{x \in H\ ;\ (x,y) = 0$ for every $y \in \mathrm{Ker}\ A\}$.

4.2. <u>Theorem</u>. Let A be a self-adjoint operator on H and let $\mathrm{Ker}_K A \neq \{0\}$.

(i) If $\xi \in A(H)$ then

$R_K(A) = A(H) \cap \{f \in H\ ;\ \omega \in \Gamma \Longrightarrow (f,\omega) > 0\}$.

(ii) If $\xi \notin A(H)$ then

$R_K(A) = \{f \in H$; there exists $a \leq 0$ such that for every $\eta \in \mathrm{Ker}\ A$ it is $(f,\eta) = a(\xi,\eta)$.

<u>Proof</u>. 1. First, suppose $f \in R_K(A)$, i.e., by Lemma 3.1, $f = Au - \beta\xi$ for $u \in K$, $\beta \geq 0$. If $\xi \in A(H)$ then $f \in A(H)$ and, moreover, for $\Gamma \neq \emptyset$ and $\omega \in \Gamma$ it is

$(f,\omega) = (Au - \beta\xi\ ,\omega) = -\beta(\xi,\omega) + (u,\xi) \geq 0$.

If $\xi \notin A(H)$ then

$(f,\eta) = (Au - \beta\xi\ ,\eta) = -\beta(\xi,\eta)$

for all $\eta \in \mathrm{Ker}\ A$, which follows the necessity part of the statement (ii).

2. Let $\xi = A\omega$ and let $f \in A(H)$, i.e. $f = A(n_0 + \gamma\xi)$, where $n_0 \in N$ and γ is real. It follows that $(f,\omega) = \gamma$ and the proof is complete in view of Lemma 3.1 if $(f,\omega) \geq 0$. In the case of $\Gamma = \emptyset$ and $(f,\omega) < 0$ it must be $\omega = n_1 + \delta\xi$ with $n_1 \in N$ and $\delta > 0$. Putting

$\beta = -\dfrac{(f, \omega)}{\delta}$ we have $f + \beta\xi = A(n_0 + \beta n_1)$,

and, by Lemma 3.1, $f \in R_K(A)$.

3. Suppose now $\xi \notin A(H) = (\mathrm{Ker}\ A)^{\perp}$ and let f satisfy the conditions of (ii). Thus there exists $\eta_0 \in \mathrm{Ker}\ A$ such that $(\xi, \eta_0) \neq 0$ and so $a = \dfrac{(f, \eta_0)}{(\xi, \eta_0)}$. Since $f - a\xi \in (\mathrm{Ker}\ A)^{\perp}$ there exists $n_0 \in N$ and real γ such that $f - a\xi = A(n_0 + \gamma\xi + r\eta_0)$ for arbitrary real r . The element η_0 can be expressed in the form $\eta_0 = n_1 + \delta\xi$ with $n_1 \in N$ and non-vanishing real δ . Therefore

$$f = A[(n_0 + rn_1) + (\gamma + r\delta)\xi] - (-a)\xi$$

and putting $r = -\dfrac{\gamma}{\delta}$ we obtain, by Lemma 3.1, $f \in R_K(A)$.

The following two remarks are the immediate consequences of the previous theorems and they will be used in Section 6.

4.3. Remark. Let Ker A be one dimensional generated by the non-trivial vector η .

(i) If $(\xi, \eta) > 0$ then $R_K(A) = \{f \in H ;\ (f, \eta) \leq 0\}$.

(ii) If $(\xi, \eta) = 0$ and $\Gamma \neq \emptyset$ then all elements of Γ are of the form $\omega + r\eta$, r arbitrary real number, and $R_K(A) = \{f \in H ;\ (f, \eta) = 0 ,\ (f, \omega) \geq 0\}$.

(iii) If $(\xi, \eta) = 0$ and $\Gamma = \emptyset$ then

$R_K(A) = \{f \in H ;\ (f, \eta) = 0\}$.

4.4. Remark. Let $\mathrm{Ker}\ A = \{0\}$ and $\Gamma \neq \emptyset$. Then

Γ contains a precisely one point ω and

$$R_K(A) = \{ f \in H ; (f, \omega) \geq 0 \} .$$

5. Nonlinear inequalities.

5.1. Theorem. Let $\mathrm{Ker}_K A = \{0\}$ and let T be a nonlinear completely continuous mapping from H into H. Suppose that there exist $c \geq 0$, $d \geq 0$, $\gamma \in \langle 0,1)$ such that

$$\| Tu \| \leq c + d \| u \|^{\gamma} \tag{5.1}$$

for all $u \in$.

Then for arbitrary $f \in H$ there exists $u \in K$ such that

$$(Au - Tu - f, v - u) \geq 0 \quad \text{for all} \quad v \in K . \tag{5.2}$$

Proof. Let $f \in H$, $\varphi \in H$. With respect to Theorem 4.1 the element $u = B(T\varphi + f)$ is the unique solution of

$$(Au - T\varphi - f, v - u) \geq 0 \quad \text{for all} \quad v \in K . \tag{5.3}$$

It is sufficient to show that the mapping $F: \varphi \longmapsto$ $\longmapsto B(T\varphi + f)$ has a fixed point in H. The mapping F is completely continuous and

$$\| F\varphi \| \leq \| A^{-1} \| (c + d \| \varphi \|^{\gamma} + \| f \|) + $$
$$+ \frac{(c + d \| \varphi \|^{\gamma} + \| f \|) \| A^{-1} \xi \|^2}{(\xi, A^{-1}\xi)} .$$

Thus there exists $R > 0$ such that $\| F\varphi \| \leq R$ for all $\varphi \in H$ with $\| \varphi \| \leq R$. The Schauder fixed point theorem implies the desired assertion.

5.2. <u>Theorem</u>. Let $\operatorname{Ker} A = \{0\}$ and let $\omega \in \Gamma$. Let T be a completely continuous operator satisfying (5.1) and let

$$\inf_{u \in H} (Tu, \omega) = c_1 > -\infty .$$

If $f \in H$, $(f,\omega) \geq -c_1$, then the variational inequality (5.2) has a solution $u \in K$.

<u>Proof</u>. It is sufficient to show that there exists $u \in K$ such that

$$u = A^{-1} Tu + A^{-1} f . \tag{5.4}$$

The Schauder fixed point theorem implies that there exists $u \in H$ satisfying (5.4). Moreover,

$$(u,\xi) = (Tu,\omega) + (f,\omega) \geq c_1 - c_1 = 0$$

and thus $u \in K$.

5.3. <u>Theorem</u>. Let $\operatorname{Ker} A = \{0\}$ and let $\omega \in \Gamma$. Suppose that T is a mapping from H into H such that

$$\sup_{u \in H} (Tu,\omega) = c_2 < +\infty .$$

Let $f \in H$, $(f,\omega) < -c_2$. Then the variational inequality (5.2) has no solution $u \in K$.

<u>Proof</u>. Suppose that (5.2) has a solution $u = n + \alpha\xi$ with $n \in N$ and $\alpha \geq 0$. From Lemma 3.1 it follows that there exists $\beta \geq 0$, $\alpha\beta = 0$, such that

$$n + \alpha\xi = A^{-1} Tu + A^{-1} f + \beta\omega .$$

If $\alpha = 0$ then

$$0 = (Tu,\omega) + (f,\omega) + \beta(\omega,\xi) < \beta(\omega,\xi) ,$$

i.e. $\beta < 0$ which is a contrary.
If $\beta = 0$ then

$$\alpha = (Tu, \omega) + (f, \omega) < c_2 - c_2 = 0 ,$$

which is again a contrary.

5.4. Theorem. Let Ker A be one-dimensional generated by the non-trivial vector η such that $(\xi, \eta) \neq 0$. Let T be a nonlinear completely continuous operator satisfying (5.1) for all $u \in N$. Let

$$\sup_{n \in N} \frac{(Tn, \eta)}{(\xi, \eta)} = c_3 < +\infty .$$

If $f \in H$, $\dfrac{(f, \eta)}{(\xi, \eta)} \leq -c_3$, then the variational inequality (5.2) has a solution $u \in K$.

Proof. It is sufficient to find $n \in N$ such that

$$(5.5) \qquad Tn + f - \frac{(Tn + f, \eta)}{(\xi, \eta)} \xi = An$$

and

$$(5.6) \qquad \frac{(Tn + f, \eta)}{(\xi, \eta)} \leq 0 .$$

Denote by L the right inverse of A from $\{\varphi \in H ; (\varphi, \eta) = 0\}$ onto N. The mapping L is continuous and the equation (5.5) has a form

$$(5.7) \qquad L\left[Tn + f - \frac{(Tn + f, \eta)}{(\xi, \eta)} \xi \right] = n .$$

The Schauder fixed point theorem implies that the equation

(5.7) (and thus also (5.5)) is solvable in N . Moreover, for arbitrary solution $n \in N$ of (5.5) it is

$$\frac{(Tn,\eta)}{(\xi,\eta)} + \frac{(f,\eta)}{(\xi,\eta)} \le c_3 - c_3 = 0$$

which verifies (5.6).

5.5. Open problems. We have not had any success to give at least sufficient conditions for the solvability of (5.2) under all other possible conditions on A considered in Section 4. Moreover, it will be interesting to give the necessary and sufficient conditions on $f \in H$ to be (5.2) solvable. For example, under assumptions of Theorems 5.2 and 5.3, the problem about the solvability of (5.2) if $-c_2 \le (f,\omega) < -c_1$, is not solved.

6. Application. Let $H = W_0^{1,2}(0,\pi)$ be the space of all absolutely continuous functions on the interval $\langle 0,\pi\rangle$ vanishing at 0 and π and derivatives of which (existing almost everywhere) are square integrable over $(0,\pi)$ in the Lebesgue sense. The space H equipped with the inner product

$$(u,v) = \int_0^\pi u'(x)\, v'(x)\, dx \ , \ u \ , \ v \in H \ ,$$

is a Hilbert space.

Let $\theta \in (0,\pi)$. Put

$$\xi(t) = \begin{cases} \dfrac{1}{\theta}\sqrt{\dfrac{\theta(\pi-\theta)}{\pi}}\, t & \text{if } 0 \le t \le \theta \\ \dfrac{\pi-t}{\pi-\theta}\sqrt{\dfrac{\theta(\pi-\theta)}{\pi}} & \text{if } \theta < t \le \pi \ . \end{cases} \tag{6.1}$$

Obviously $\xi \in H$, $\|\xi\| = 1$, and $K = \{v \in H , v(\theta) \geq 0\}$.

Let λ be a real number. Define the mapping A_λ : $H \to H$ such that for all $u , v \in H$ it is

$$(A_\lambda u,v) = \lambda \int_0^\pi u'(x) v'(x) dx - \int_0^\pi u(x) v(x) dx .$$

It is easy to see that A_λ satisfies all assumptions of Section 2 and, moreover,

Ker $A_\lambda = \{0\}$ if and only if $\lambda \neq \lambda_j = \frac{1}{j^2}$, j is a positive integer,

and, if $\lambda = \lambda_j$ then Ker A_λ is a linear hull of sin jx .

Let $f \in L_1 (0,\pi)$. Then

$$\varphi(v) = \int_0^\pi f(x) v(x) dx , v \in H ,$$

is a bounded linear functional on H . Thus, in view of the Riesz representation theorem, there exists the uniquely determined $Uf \in H$ such that $(Uf, v) = \varphi(v)$ for all $v \in H$.

Put

$$\mathcal{R}_K(A_\lambda) = U^{-1} (R_K (A_\lambda)) ,$$

i.e. $f \in L_1 (0,\pi)$ is an element of $\mathcal{R}_K (A_\lambda)$ if and only if there exists at least one $u \in K$ such that

(6.2) $$\lambda \int_0^\pi u'(x) [v'(x) - u'(x)] dx - \int_0^\pi u(x) [v(x) - u(x)] dx \geq \int_0^\pi f(x) [v(x) - u(x)] dx$$

for all $v \in K$.

6.1. Lemma. Let $f \in C\langle 0 , \pi\rangle \cap \mathcal{R}_K (A_\lambda)$ and let

$u \in K$ be a solution of (6.2).

(i) If $u(\theta) > 0$ then $u \in C^2\langle 0, \pi\rangle$, and

$$- \lambda u''(x) - u(x) = f(x)$$

for all $x \in \langle 0, \pi\rangle$.

(ii) If $u(\theta) = 0$ then $u \in C^2\langle 0, \theta\rangle \cap C^2\langle \theta, \pi\rangle$, and

$$- \lambda u''(x) - u(x) = f(x)$$

for all $x \in \langle 0, \theta) \cap (\theta, \pi\rangle$ and $\lambda u'(\theta-) \geq \lambda u'(\theta+)$.
Conversely, if $f \in C\langle 0, \pi\rangle$ and u satisfies either the conditions (i) or (ii) then $u \in K$ and u is a solution of (6.2).

Proof. 1. First, suppose that $u \in K$ is a solution of (6.2). In the case of $u(\theta) > 0$ the function u belongs to K^0 which means that $A_\lambda u = Uf$. If $u(\theta) = 0$ then (see Lemma 3.1) $Uf = A_\lambda u - \beta\xi$ with some non negative β. Using integration by parts we obtain either

$$\int_0^\pi [\lambda u'(x) + \int_0^x u(t)\, dt + \int_0^x f(t)\, dt]\ \varphi'(x)\, dx = 0$$

or

$$\int_0^\pi [\lambda u'(x) + \int_0^x u(t)\, dt - \beta\xi'(x) + \int_0^x f(t)\, dt]\ \varphi'(x) dx = 0$$

for all $\varphi \in H$. Using the standard regularity argument we prove that u satisfies the differential equation either on $\langle 0, \pi\rangle$ (in the case of (i)) or on $\langle 0, \theta) \cup (\theta, \pi\rangle$ (in the case of (ii)). Further, by the second equality we obtain

$$\lambda [u'(\theta-) - u'(\theta+)] = \beta\sqrt{\frac{\pi}{\theta(\pi-\theta)}} \geq 0 .$$

It follows the last statement of (ii).

2. Conversely, if $u \in H$ satisfies either the conditions (i) or (ii) with $f \in C\langle 0, \pi \rangle$ then it immediately follows that $u \in K$ and $A_\lambda u = Uf$ in the case of (i). If the conditions (ii) are fulfilled we have, using integration by parts,

$$- \lambda \varphi(\theta) [u'(\theta-) - u'(\theta+) + (A_\lambda u - Uf, \varphi) = 0$$

for all $\varphi \in H$. Taking $\varphi = v - u$, $v \in K$, the assumptions $\lambda u'(\theta-) \geq \lambda u'(\theta+)$ and $u(\theta) = 0$ lead to the inequality

$$(A_\lambda u - Uf, v - u) \geq 0$$

holding for every $v \in K$. By definition, it follows $f \in \mathcal{R}_K(A_\lambda)$.

Similarly one can prove

6.2. <u>Lemma</u>. Let $\omega \in H$ satisfy $A_\lambda \omega = \xi$ (ξ is given by (6.1)). Then $\omega \in C^2\langle 0, \pi \rangle$ and

$$- \lambda \omega''(x) - \omega(x) = \xi(x)$$

for all $x \in \langle 0, \pi \rangle$.

Keeping the notation introduced in Sections 2,3 and solving of the equation from Lemma 6.2 it is possible to prove by computation the following three lemmas.

6.3. <u>Lemma</u>. Let $\lambda = -\frac{1}{\alpha^2}$, $\alpha > 0$. Then it is always $\Gamma = \{\omega\}$, where

(6.3a) $\omega(t) = \sqrt{\frac{\theta(\pi - \theta)}{\pi}} \left\{ - \frac{t}{\theta} + \frac{\pi}{\alpha\theta(\pi - \theta)} [\sinh \alpha(t - \theta) + \right.$

$$+ \quad e^{-\alpha t} \quad \frac{1 - e^{2\alpha(t-\pi)}}{1 - e^{-2\alpha\pi}} \quad \sinh \alpha\theta] \Big\}$$

if $0 \leq t \leq \theta$ and

(6.3b) $\omega(t) =$

$$= \sqrt{\frac{\theta(\pi-\theta)}{\pi}} \left\{ \frac{t-\pi}{\pi-\theta} + \frac{\pi}{\alpha\theta(\pi-\theta)} e^{-\alpha t} \frac{1-e^{2\alpha(t-\pi)}}{1-e^{-2\alpha\pi}} \sinh \alpha\theta \right\}$$

if $\theta < t \leq \pi$.

6.4. Lemma. Let $\lambda = \frac{1}{\alpha^2}$, $\alpha > 0$, α is not a positive integer.

(i) If

$$\frac{\pi}{\theta(\pi-\theta)} \quad \frac{\sin \alpha\theta \,.\, \sin \alpha(\pi-\theta)}{\alpha \sin \alpha\pi} \leq 1 \tag{6.4}$$

then $\Gamma = \{\omega\}$, where

$$\text{(6.5a)} \quad \omega(t) = \sqrt{\frac{\theta(\pi-\theta)}{\pi}} \left\{ -\frac{t}{\theta} + \frac{\pi}{\theta(\pi-\theta)} \frac{\sin \alpha(\pi-\theta)}{\alpha \sin \alpha\pi} \sin \alpha t \right\}$$

if $0 \leq t \leq \theta$ and

(6.5b) $\omega(t) =$

$$= \sqrt{\frac{\theta(\pi-\theta)}{\pi}} \left\{ \frac{t-\pi}{\pi-\theta} + \frac{\pi}{\theta(\pi-\theta)} \quad \frac{\sin \alpha\theta}{\alpha \sin \alpha\pi} \sin \alpha(\pi-t) \right\}$$

if $\theta < t \leq \pi$.

(ii) If

$$\frac{\pi}{\theta(\pi-\theta)} \quad \frac{\sin \alpha\theta \,.\, \sin \alpha(\pi-\theta)}{\alpha \sin \alpha\pi} > 1 \tag{6.6}$$

then $\Gamma = \emptyset$.

6.5. Lemma. Let $\lambda = \lambda_j = \frac{1}{j^2}$ for some positive integer j .

(i) If $\Theta = \frac{k}{j}\pi$, $k = 1, \dots, j-1$, then $\Gamma = \{\omega + a \sin jt \, ; \, a \text{ real}\}$, where

$$\omega(t) = \left(-t + \frac{(-1)^k}{j-k} \sin jt\right) \sqrt{\frac{j-k}{k\pi}} \tag{6.7a}$$

if $0 \le t \le \frac{k}{j}\pi$ and

$$\omega(t) = (t-\pi) \sqrt{\frac{k}{\pi(j-k)}} \tag{6.7b}$$

if $\frac{k}{j}\pi < t \le \pi$.

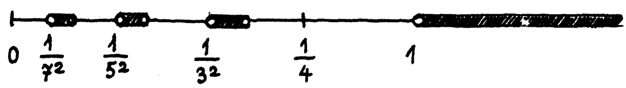
(ii) If $\Theta \in (0,\pi) \setminus \left\{\frac{\pi}{j}, \dots, \frac{j-1}{j}\pi\right\}$ then $\Gamma = \emptyset$.

Now, we are ready to apply the abstract results of Section 4.

6.6. Theorem. It is $\mathcal{R}_K(A_\lambda) = L_1(0,\pi)$ if and only if $\lambda = \frac{1}{\alpha^2}$, $\alpha > 0$, α is not an integer and the inequality (6.6) is fulfilled.

6.7. Remark. Especially, from Theorem 5.6 it can be shown that $\mathcal{R}_K(A_\lambda) = L_1(0,\pi)$ provided $\lambda > 1$. This result is also possible to derive from the abstract theorem in [1] mentioned in Section 1.

6.8. Remark. For $\Theta = \frac{\pi}{2}$ the infinitely many intervals of λ for which $\mathcal{R}_K(A_\lambda) = L_1(0,\pi)$ can be found from Lemma 6.4 and Theorem 6.6 as it is sketched in the following figure:



The situations for other $\theta \in (0,\pi)$ are similar but they are more complicated.

6.9. __Theorem.__ (i) $\mathcal{R}_K(A_\lambda) = \{f \in L_1(0,\pi)\ ;\ \int_0^\pi f(x)\,\omega(x)\,dx \geq 0\}$ if and only if either $\lambda < 0$ and ω is given by (6.3) or $\lambda > 0$, $\lambda \neq \frac{1}{j^2}$, $j = 1,\ldots$, (6.4) holds and ω is given by (6.5).

(ii) $\mathcal{R}_K(A_\lambda) = \{f \in L_1(0,\pi)\ ;\ \int_0^\pi f(x)\sin jx\,dx \leq 0\}$ if and only if $\lambda = \frac{1}{j^2}$, j is a positive integer, and $\theta \in \left(\frac{2k}{j}\pi, \frac{2k+1}{j}\pi\right)$, $k = 0,\ldots,[\frac{j-1}{2}]$.

(iii) $\mathcal{R}_K(A_\lambda) = \{f \in L_1(0,\pi)\ ;\ \int_0^\pi f(x)\sin jx\,dx \geq 0\}$ if and only if $\lambda = \frac{1}{j^2}$, j is a positive integer, and $\theta \in \left(\frac{2k+1}{j}\pi, \frac{2k+2}{j}\pi\right)$, $k = 0,\ldots,[\frac{j}{2}] - 1$.

(iv) $\mathcal{R}_K(A_\lambda) = \{f \in L_1(0,\pi)\ ;\ \int_0^\pi f(x)\sin jx\,dx = 0$ and $\int_0^\pi f(x)\,\omega(x)\,dx \geq 0\}$ if and only if $\lambda = \frac{1}{j^2}$, j is a positive integer, $\theta = \frac{k}{j}\pi$, $k = 1,\ldots, j-1$, and ω is given by (6.7).

Some results about the solvability of nonlinear inequalities with ordinary differential operators are also possible to derive on the base of Section 5 and the previous lemmas.

6.10. <u>Theorem</u>. Let g be a continuous real valued function defined on the real line. Suppose that there exist $c \geq 0$, $d \geq 0$, $\gamma \in \langle 0,1)$ such that

$$|g(\xi)| \leq c + d|\xi|^{\gamma} \tag{6.8}$$

for all real ξ .

Then the variational inequality

$$\lambda \int_0^{\pi} u'(x)\,[v'(x) - u'(x)]\,dx - \int_0^{\pi} u(x)\,[v(x) - u(x)]\,dx + \int_0^{\pi} g(u(x))\,[v(x) - u(x)]\,dx \geq \int_0^{\pi} f(x)\,[v(x) - u(x)]\,dx \quad \text{for all } v \in K \tag{6.9}$$

has a solution $u \in K$ for arbitrary $f \in L_1(0,\pi)$ provided that $\lambda = \frac{1}{\alpha^2}$, $\alpha > 0$, α is not an integer and (6.6) holds.

6.11. <u>Theorem</u>. Let $\lambda = \frac{1}{j^2}$, j is a positive integer, and $\theta \in \left(\frac{2k}{j}\pi, \frac{2k+1}{j}\pi\right)$, $k = 0,\dots,\left[\frac{j-1}{2}\right]$. Let g be a continuous real valued function defined on the real line and

$$\sup_{\xi \in \mathbb{R}} |g(\xi)| = c < +\infty . \tag{6.10}$$

Then the variational inequality (6.9) has a solution u K provided that $f \in L_1(0,\pi)$ and

$$\int_0^{\pi} f(x) \sin jx \, dx \leq -2c .$$

6.12. <u>Remark</u>. An analogous assertion as that in Theorem 6.12 is also possible to be given for

$$\theta \in \left(\frac{2k+1}{j} \pi , \frac{2k+2}{j} \pi \right), \quad k = 0, \dots, \left[\frac{j}{2} \right] - 1 .$$

6.13. <u>Theorem</u>. Suppose (6.10). Then the variational inequality (6.9) has a solution $u \in K$ provided that $f \in L_1(0,\pi)$ and one of the following conditions is fulfilled:

(i) $\lambda = -\frac{1}{\alpha^2}$, $\alpha > 0$, and

$$\int_0^{\pi} f(x)\, \omega(x)\, dx \geq c \int_0^{\pi} |\omega(x)|\, dx , \tag{6.11}$$

where ω is given by (6.3) ;

(ii) $\lambda = \frac{1}{\alpha^2}$, $\alpha > 0$, α is not an integer, (6.4) and (6.11) hold, where ω is given by (6.5).

R e f e r e n c e

[1] J.L. LIONS: Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris 1969.

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