Commentationes Mathematicae Universitatis Carolinae

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*Commentationes Mathematicae Universitatis Carolinae*, Vol. 16 (1975), No. 4, 637--640

Persistent URL: http://dml.cz/dmlcz/105653

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,4 (1975)

# REAL CLASSES IN THE ULTRAPOWER OF HEREDITARILY FINITE SETS

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Abstract: For every non-standard $n^*$ we construct $F$ such that in the ultrapower of the set of all hereditarily finite sets it holds: " $F$ is a function from $n^*$ into a cofinal part of On and for every set $x$ the intersection $F \cap x$ is a set."

Key words: ultrapower, model, non-standard natural number.

AMS: Primary: 02H05, 02H20, 10N10 Ref. Ž.: 2.666, 1.99
Secondary: 02K05, 02K10

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The model of all hereditarily finite sets ( $\mathfrak{N} = \langle V_\omega, E \upharpoonright V_\omega \rangle$ ) is a model of ZF set theory of finite sets ($ZF_{Fin}$). The same is true for the ultrapower $\mathfrak{M} = \mathfrak{N}^\omega / _Z$ ( $= \langle M, \widetilde{E} \rangle$ , say; $Z$ is supposed to be a non-trivial ultrafilter on $\omega$ ). If we add to $\mathfrak{N}$ all subsets of $V_\omega$ we obtain a model of GB set theory of finite sets ($GB_{Fin}$) . Let $\mathfrak{M}'$ denote the model obtained from $\mathfrak{M}$ by adding all subsets of $M$ (i.e. $\mathfrak{M}' = \langle M \cup Q, \widetilde{E} \cup E \upharpoonright Q \rangle$ where $Q = \{x \in P(M);\ \neg (\exists f)(x = \{g;\ \mathfrak{M} \models g \in f\})\}$). Now the situation is quite different (we have $\mathfrak{M}' \not\models GB_{Fin}$) since there is a "finite cofinal part of On "; more precisely for every non-standard natural number $n^*$ it holds in $\mathfrak{M}'$ that there is a mapping from $n^*$ onto a cofinal part of On .

We say that a class $X$ is real ( Real $(X)$ , see 1402 [1]) if the intersection of $X$ with every set is a set. Our question is if the submodel of $\mathfrak{M}'$ consisting of $\mathfrak{M}'$-real classes is a model of $GB_{Fin}$ . The answer is negative since we shall construct a "real finite cofinal part of On ". Consequently real classes of $\mathfrak{M}'$ cannot be closed under gödelian operations.

This paper arose in connection with building up of the Alternative Theory of Sets (see [2]). The inconsistency of some strengthening of the axioms of this theory was shown by using the construction described in the paper.

We shall suppose AC and CH .

<u>Remark</u>. Ultrapower as syntactical model ($*$ , say) is an interpretation of GB in GB . Our construction shows that there is $X \subseteq \omega^*$ such that the intersection of $X$ with every $*$-natural number is a $*$-set and $X$ itself is not a $*$-set. In other words, there is an interpretation of the theory of semisets with the axiom $(\exists X \subseteq \omega)(\forall n \in \omega)(M(X \cap n) \& \neg M(X))$ in GB .

<u>Lemma</u>. If $\varphi$ is a ZF-formula and if $X \subseteq M$ is countable then $\mathfrak{M}' \models (\forall x \subseteq X)\varphi(x) \longrightarrow (\exists y)(\varphi(y) \& X \subseteq y)$ .

<u>Proof</u>: Let $X = \{g_i ; i \in \omega\}$ & $\mathfrak{M}' \models (\forall \dot{x} \subseteq X)\varphi(x)$ and let $k_x$ denote the constant the value of which is $x$ . Put $An = \{k_n ; n \in \omega\}$ (the class of all standard natural numbers). Let us define $f(n) = \{\langle g_i(n), i\rangle ; i \leq n\}$ . Then for every $n \in \omega$ we have

(1) $\mathfrak{M}' \models X = f'' An \& \varphi(f'' k_n)$ .

Since $\mathfrak{M}$ is a model of $ZF_{Fin}$ there is $n^*$ the minimal natural number (less than diagonal, say) with

$$\mathfrak{M} \models \varphi(f'' n^*) \ .$$

According to (1) we have $\mathfrak{M}' \models X \subseteq f'' n^*$ .

<u>Theorem</u>. For every $n^*$ non-standard natural number of $\mathfrak{M}$ there is $F$ with

$\mathfrak{M}' \models \mathrm{Real}(F)$ & $\mathrm{Fnc}(F)$ & $D(F) \subseteq n^*$ & $(\forall k)(\exists \ell)(\ell \in W(F)$ & & $k < \ell)$.

<u>Proof</u>: Let $n^*$ be given, let $\{g_\alpha ; \alpha \in \aleph_1\}$ be a monotonous part of $n^*$ (i.e. $\alpha < \beta < \aleph_1 \longrightarrow \mathfrak{M} \models g_\alpha \in g_\beta \in$ $\in n^*$) and let $\{h_\alpha ; \alpha \in \aleph_1\}$ be a monotonous cofinal part of $On^{\mathfrak{M}}$ (i.e. $\alpha < \beta < \aleph_1 \longrightarrow . \mathfrak{M} \models h_\alpha \in h_\beta \in On$ & & $(\forall h)(\exists \alpha \in \aleph_1) \mathfrak{M} \models h \in On \longrightarrow h \in h_\alpha)$ . We define by induction the sequence $\{f_\alpha ; \alpha \in \aleph_1\}$ such that

(2) $\mathfrak{M} \models f_0 = 0$

(3) $\mathfrak{M} \models f_{\alpha+1} = f_\alpha \cup \{\langle h_\alpha \, g_\alpha \rangle\}$

(4) $\alpha$ limit & $\beta < \alpha . \longrightarrow \mathfrak{M} \models f_\beta \subseteq f_\alpha$ & " $f_\alpha$ is a monotonous function from $g_\alpha$ into $h_\alpha$ ".

The existence of $f_\alpha$ for limit $\alpha$ follows from the induction hypothesis and from Lemma (put $X = \{f_\beta ; \beta \in \alpha\}$ , $\varphi$ denotes "$\bigcup x$ is a monotonous function from $g_\alpha$ into $h_\alpha$ " and put $f_\alpha = \bigcup y$ ). Finally let $\mathfrak{M}' \models$ " $F = \bigcup\{f_\alpha ; \alpha \in \aleph_1\}$ " i.e. $(\forall \alpha \in \aleph_1)\ \mathfrak{M}' \models f_\alpha \subseteq F$ & $(\forall x)(\exists \alpha)\ \mathfrak{M}' \models x \in F \longrightarrow$ $\longrightarrow x \in f_\alpha$ . Now we have

1) $\mathfrak{M}' \models$ " $F$ is a monotonous function from $n^*$ into $On$ " (by (4)).

2) $(\forall \alpha \in \aleph_1)\ \mathfrak{M}' \models h_\alpha \in W(F)$ (by (3)) and therefore values of $F$ form a cofinal part of $On^{\mathfrak{M}}$ .

3) $\mathfrak{M}' \models \mathrm{Real}(F)$ . Let $x \in M$ then there is $\alpha$ such that $\mathfrak{M} \models W(x) \cap On \subseteq h_\alpha$ . By using monotony of $f_\beta$'s we have $\mathfrak{M}' \models (F - f_\alpha) \cap x = 0$ and therefore $\mathfrak{M}' \models F \cap x = f_\alpha \cap x$ . Since $\mathfrak{M} \models M(f_\alpha \cap x)$ it is $\mathfrak{M}' \models M(F \cap x)$ .

R e f e r e n c e s

[1] P. VOPĚNKA and P. HÁJEK: The theory of semisets, North-Holland P.C. and Academia, Prague, 1972.

[2] P. VOPĚNKA: Matematika v alternativní teorii množin (Mathematics in the Alternative theory of sets), manuscript.

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(Oblatum 30.5.1975)