Commentationes Mathematicae Universitatis Carolinae

Josef Jirásko

Generalized injectivity

Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 4, 621--636

Persistent URL: http://dml.cz/dmlcz/105652

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

16,4 (1975)

GENERALIZED INJECTIVITY

J. JIRÁSKO, Praha

Abstract: In this paper, a new general theory of injectivity of left R-modules is introduced. The existence and unicity of the injective envelope of every module is established for a large class of injectivities. Some earlier known results on injectivities with respect to preradicals are derived from the theory in a more general form.

Key-words: $\mathcal{L}$-injective module, $\mathcal{L}$-injective envelope, preradical.

AMS: Primary 16A52, 16A64, Ref. Ž.: 2.723
Secondary: 18E40

We start with some basic definitions and notations. Throughout this paper, R stands for an associative ring with unit element and R-mod denotes the category of all unitary left R-modules. If $f \in Hom_R(N,M)$ and P is a submodule of M then $f^{-1}(P) = \{x \in N, f(x) \in P\}$. The fact that A is an essential submodule of B (i.e. A meets every nonzero submodule of B in a nonzero submodule) will be denoted by $A \subseteq' B$.

A class $\mathcal{A}$ of modules is said to be abstract if it is closed under isomorphisms, hereditary if it is abstract and closed under submodules and cohereditary if it is closed under homomorphic images.

§ 1. <u>General theory</u>. In the class $\mathfrak{M} = \{\langle M,N,f,Q\rangle,\ M,N,Q \in R\text{-mod},\ N \subseteq M,\ f \in \mathrm{Hom}_R(N,Q)\}$ define the partial order $\leq$ in the following way:

$\langle M,N,f,Q\rangle \leq \langle M',N',f',Q'\rangle$ if and only if $M = M'$, $N \subseteq N'$, $Q = Q'$ and $f'|_N = f$.

In this paragraph $\mathcal{L}$ always denotes a subclass of $\mathfrak{M}$. The following five conditions on $\mathcal{L}$ will be useful later.

(α) $\langle M,N,f,Q\rangle \in \mathcal{L}$, $\langle M,N',f',Q\rangle \in \mathfrak{M}$, $\langle M,N,f,Q\rangle \leq \langle M,N',f',Q\rangle$ implies $\langle M,N',f',Q\rangle \in \mathcal{L}$,

(β) $\langle M,N,f,A\rangle \in \mathcal{L}$, $A \overset{i}{\hookrightarrow} B$ implies $\langle M,N,if,B\rangle \in \mathcal{L}$,

(β') $\langle M,N,f,A\rangle \in \mathcal{L}$, $A \overset{i}{\hookrightarrow} B$, $A \subseteq' B$ implies $\langle M,N,if,B\rangle \in \mathcal{L}$,

(γ) $\langle M,N,f,A\rangle \in \mathcal{L}$, $A \xrightarrow{g} B$ an isomorphism, implies $\langle M,N,gf,B\rangle \in \mathcal{L}$,

(δ) $\langle M,N,f,A\rangle \in \mathcal{L}$, $A \xrightarrow{g} B$ implies $\langle M,N,gf,B\rangle \in \mathcal{L}$.

For every $\langle B,A,f,Q\rangle \in \mathfrak{M}$ let us define $r_{\mathcal{L}}(B,A,f,Q)$ ($s_{\mathcal{L}}(B,A,f,Q)$) to be a submodule of B generated by all the $g(M)$, $g \in \mathrm{Hom}_R(M,B)$, to which there exists a commutative diagram

$$\begin{array}{ccc} N & \hookrightarrow & M \\ h\downarrow & & \downarrow g \\ A & \hookrightarrow & B \\ f\downarrow & & \\ Q & & \end{array}$$

with $\langle M,N,fh,Q\rangle \in \mathcal{L}$ ($N = g^{-1}(A)$).

We also use the following abbreviations: $r_{\mathcal{L}}(B,Q,1_Q,Q) = r_{\mathcal{L}}(B,Q)$, $s_{\mathcal{L}}(B,Q,1_Q,Q) = s_{\mathcal{L}}(B,Q)$, $r_{\mathcal{L}}(\hat{Q},Q) = r_{\mathcal{L}}(Q)$, $s_{\mathcal{L}}(\hat{Q},Q) = s_{\mathcal{L}}(Q)$.

Lemma 1.1. Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ and

$$
(*)\qquad \begin{array}{ccccc}
Q & \xleftarrow{\ f\ } & A & \hookrightarrow & B \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l'} & & \downarrow{\scriptstyle l} \\
T & \xleftarrow{\ h\ } & C & \hookrightarrow & D
\end{array}
\qquad (**)\qquad \begin{array}{ccc}
P & \hookrightarrow & M \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\
A & \hookrightarrow & B
\end{array}
$$

commutative diagrams. If for every diagram $(**)$, $\langle M,P,kfg,T\rangle \in \mathcal{L}$ whenever $\langle M,P,fg,Q\rangle \in \mathcal{L}$ then $l(r_{\mathcal{L}}(B,A,f,Q)) \subseteq r_{\mathcal{L}}(D,C,h,T)$. Especially, $l(r_{\mathcal{L}}(B,Q)) \subseteq$ $\subseteq r_{\mathcal{L}}(D,T)$.

Proof: Obvious.

Definition 1.2. We say that a module Q is $\mathcal{L}$-injective, if every diagram

$$
(1)\qquad \begin{array}{ccc}
N & \hookrightarrow & M \\
{\scriptstyle f}\downarrow & & \\
Q & &
\end{array}
$$

with $\langle M,N,f,Q\rangle \in \mathcal{L}$ can be completed to a commutative one.

Theorem 1.3. Consider the following five conditions concerning a module Q :

(i) Q is a direct summand in each extension $N \supseteq Q$ such that $N \subseteq Q + r_{\mathcal{L}}(\hat{N},Q)$,

(ii) $Q \supseteq r_{\mathcal{L}}(Q)$,

(iii) every diagram (1) with $M \subseteq N + r_{\mathcal{L}}(\hat{M},N,f,Q)$ can be made commutative,

(iv) Q is $\mathcal{L}$-injective,

(v) $Q \supseteq s_{\mathcal{L}}(Q)$.

Then the conditions (i),(ii),(iii) are equivalent and (iii) implies (iv) and (iv) implies (v). Moreover, if $\mathcal{L}$ satisfies (α) then all the five conditions are equivalent.

Proof: (i) implies (ii). The module $N = Q + r_{\mathcal{L}}(Q)$ is an essential extension of Q so that $N = Q$ by (i).

(ii) implies (iii). Consider the diagram (1) and extend f to $g: \hat{M} \longrightarrow \hat{Q}$. Then $g(M) \subseteq g(N) + g(r_{\mathcal{L}}(\hat{M},N,f,Q)) \subseteq Q + r_{\mathcal{L}}(Q) = Q$.

(iii) implies (i). Obvious.

(iv) implies (v). Take the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\ i\ } & M \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Q & \xrightarrow{\ j\ } & \hat{Q} \end{array}$$

where $N = g^{-1}(Q)$ and $\langle M,N,h,Q \rangle \in \mathcal{L}$.

Then $h = fi$ for some $f: M \longrightarrow Q$ and $g = jf$ since $(g - jf)(M) \cap Q = 0$, as it is easily seen.

(ii) implies (iv). Obvious.

Finally, if $\mathcal{L}$ satisfies (α) then $r_{\mathcal{L}} = s_{\mathcal{L}}$ and we are through.

Corollary 1.4: Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfying (β). If $P, Q \in R\text{-mod}$, $P \subseteq Q$, then P is $\mathcal{L}$-injective provided $P \supseteq r_{\mathcal{L}}(Q)$.

Proof: We have $r_{\mathcal{L}}(P) \subseteq r_{\mathcal{L}}(Q)$ by Lemma 1.1 and P is $\mathcal{L}$-injective by Theorem 1.3.

Definition 1.5. For any ordinal α and modules $A \subseteq B$ let us define the sequence $r_{\mathcal{L}}^{\alpha}(\hat{B},A)$ of modules inductively as follows:

$$r_{\mathcal{L}}^{0}(\hat{B},A) = A$$

$$r_{\mathcal{L}}^{\alpha+1}(\hat{B},A) = r_{\mathcal{L}}^{\alpha}(\hat{B},A) + r_{\mathcal{L}}(\hat{B}, r_{\mathcal{L}}^{\alpha}(\hat{B},A))$$

and $r_{\mathcal{L}}^{\alpha}(\hat{B},A) = \bigcup_{\beta<\alpha} r_{\mathcal{L}}^{\beta}(\hat{B},A)$, α limit.

Further, put $\bar{r}_{\mathcal{L}}(\hat{B},A) = r_{\mathcal{L}}^{\alpha}(\hat{B},A)$ where $r_{\mathcal{L}}^{\alpha}(\hat{B},A) = r_{\mathcal{L}}^{\alpha+1}(\hat{B},A)$.

Corollary 1.6: For every module Q , the module $\bar{r}_{\mathcal{L}}(\hat{Q},Q)$ is $\mathcal{L}$-injective.

Proof: Denote $P = \bar{r}_{\mathcal{L}}(\hat{Q},Q)$. Then $r_{\mathcal{L}}(P) = r_{\mathcal{L}}(\hat{Q},P) \subseteq \bar{r}_{\mathcal{L}}(\hat{Q},Q) = P$ and apply Theorem 1.3.

Lemma 1.7. Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfying (γ) and $A \subseteq B$ be modules. If $f: \hat{B} \longrightarrow \tilde{B}$ is a B-isomorphism of two injective envelopes $\hat{B}, \tilde{B}$ of B then $f(r_{\mathcal{L}}^{\alpha}(\hat{B},A)) = r_{\mathcal{L}}^{\alpha}(\tilde{B},A)$.

Proof: It follows easily by transfinite induction using Lemma 1.1.

Lemma 1.8. Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfiyng (β'). If $Q \subseteq P \subseteq \hat{Q}$ then $r_{\mathcal{L}}^{\alpha}(\hat{Q},Q) \subseteq r_{\mathcal{L}}^{\alpha}(\hat{Q},P)$ for every ordinal α .

Proof: By transfinite induction and Lemma 1.1.

Theorem 1.9. Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfying $(\alpha),(\beta')$. Then $\bar{r}_{\mathcal{L}}(\hat{Q},Q)$ is the smallest $\mathcal{L}$-injective submodule of $\hat{Q}$ containing Q .

Proof: The module $\bar{r}_{\mathcal{L}}(\hat{Q},Q)$ is $\mathcal{L}$-injective by Corollary 1.6. Let a module P , $Q \subseteq P \subseteq \hat{Q}$, be $\mathcal{L}$-injective. From Theorem 1.3 we get $P = r_{\mathcal{L}}^{1}(\hat{Q},P)$ and Lemma 1.8 then yields $\bar{r}_{\mathcal{L}}(\hat{Q},Q) \subseteq P$.

Definition 1.10. An $\mathcal{L}$-injective module B is said to be an $\mathcal{L}$-injective envelope of a module A if there is no proper $\mathcal{L}$-injective submodule of B containing A .

Remark 1.11: If $\mathcal{L}$ satisfies (γ) then the class of $\mathcal{L}$-injective modules is abstract.

Theorem 1.12. If the subclass $\mathcal{L}$ of $\mathcal{M}$ satisfies

$(\alpha),(\beta)$ and (γ) then every module Q has an $\mathcal{L}$-injective envelope which is unique up to Q-isomorphism.

__Proof__: The existence of an $\mathcal{L}$-injective envelope of Q follows from Theorem 1.9. Put $Q_\alpha = r_{\mathcal{L}}^{\alpha}(\hat{Q},Q)$ and let f_0 be the canonical embedding $Q \hookrightarrow P$ where P is an arbitrary $\mathcal{L}$-injective envelope of Q. Suppose that $f_\alpha : Q_\alpha \to P$ is a monomorphism. From $Q_{\alpha+1} = Q_\alpha + r_{\mathcal{L}}(\hat{Q},Q_\alpha)$ we get $Q_{\alpha+1} \subseteq Q_\alpha + r_{\mathcal{L}}(\hat{Q},Q_\alpha, f_\alpha, P)$ by Lemma 1.1 (using $(\beta),(\gamma)$) and consequently f_α extends to a homomorphism $f_{\alpha+1}: Q_{\alpha+1} \to P$ by Theorem 1.3. It is easy to see that $f_{\alpha+1}$ is a monomorphism. From this, one easily derives the existence of a monomorphism $f: \overline{Q} = \overline{r}_{\mathcal{L}}(\hat{Q},Q) \to P$ extending the identity on Q. Hence $f(\overline{Q})$ is $\mathcal{L}$-injective by 1.11, so that $f(\overline{Q}) = P$, which finishes the proof.

__Definition 1.13.__ A submodule A of a module B is said to be $\mathcal{L}$-dense in B if $B \subseteq A + r_{\mathcal{L}}(\hat{B},A)$. An essential $\mathcal{L}$-dense submodule A of B is said to be $\mathcal{L}$-essential.

__Theorem 1.14.__ If $\mathcal{L}$ satisfies (α) then a module Q is $\mathcal{L}$-injective if and only if it has no proper $\mathcal{L}$-essential extensions.

__Proof__: The condition is necessary since every $\mathcal{L}$-dense extension of an $\mathcal{L}$-injective module splits by Theorem 1.3. Conversely, $r_{\mathcal{L}}^{1}(\hat{Q},Q)$ is an $\mathcal{L}$-essential extension of Q so that $r_{\mathcal{L}}(Q) \subseteq Q$ and Q is $\mathcal{L}$-injective by Theorem 1.3.

__Definition 1.15.__ Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfying (γ). A module A is said to be weakly $\mathcal{L}$-dense in B if $B \subseteq \bar{F}_{\mathcal{L}}(\hat{B},A)$. An essential weakly $\mathcal{L}$-dense submodule A of B is said to be weakly $\mathcal{L}$-essential.

Theorem 1.16. If $\mathcal{L}$ satisfies (α) and (γ) then a module Q is $\mathcal{L}$-injective if and only if it has no proper weakly $\mathcal{L}$-essential extensions.

Proof: The sufficiency follows from Theorem 1.14. Conversely, suppose that K is a weakly $\mathcal{L}$-essential extension of Q. Then $r^1_{\mathcal{L}}(\hat{Q},Q) = Q$ by Theorem 1.3 and so $Q \subseteq K \subseteq \bar{F}_{\mathcal{L}}(\hat{K},Q) = \bar{F}_{\mathcal{L}}(\hat{Q},Q) = Q$.

Remark 1.17: If $\mathcal{L}$ satisfies (γ) then $\bar{F}_{\mathcal{L}}(\hat{Q},Q)$ is the greatest weakly $\mathcal{L}$-dense extension of Q contained in $\hat{Q}$ since for a weakly $\mathcal{L}$-dense extension P of Q with $Q \subseteq P \subseteq \hat{Q}$ we have $P \subseteq \bar{F}_{\mathcal{L}}(\hat{P},Q) = \bar{F}_{\mathcal{L}}(\hat{Q},Q)$.

Lemma 1.18. Let $\mathcal{L}$ satisfy (β') and (γ) and $A \subseteq B \subseteq C$ be modules. Then

(i) if A is a weakly $\mathcal{L}$-essential submodule of C then B is weakly $\mathcal{L}$-essential in C,

(ii) if A is weakly $\mathcal{L}$-essential in B and B weakly $\mathcal{L}$-essential in C then A is weakly $\mathcal{L}$-essential in C.

Proof: (i) is immediate since Lemma 1.1 yields $\bar{F}_{\mathcal{L}}(\hat{C},A) \subseteq \bar{F}_{\mathcal{L}}(\hat{C},B)$. Further, we have $B \subseteq \bar{F}_{\mathcal{L}}(\hat{B},A)$, $C \subseteq \bar{F}_{\mathcal{L}}(\hat{C},B)$ and Lemma 1.1 gives $C \subseteq \bar{F}_{\mathcal{L}}(\hat{C},B) \subseteq \bar{F}_{\mathcal{L}}(\hat{C},\bar{F}_{\mathcal{L}}(\hat{C},A)) = \bar{F}_{\mathcal{L}}(\hat{C},A)$.

Theorem 1.19. The following are equivalent for a class $\mathcal{L}$ satisfying (α), (β) and (γ):

(i) N is a maximal weakly $\mathcal{L}$-essential extension of Q,

(ii) N is an $\mathcal{L}$-injective envelope of Q,

(iii) N is $\mathcal{L}$-injective weakly $\mathcal{L}$-essential extension of Q.

Proof: (i) implies (ii). The $\mathcal{L}$-injectivity of N follows from 1.16. If K is $\mathcal{L}$-injective, $Q \subseteq K \subseteq N$ then 1.18 and 1.16 yield $K = N$.

(ii) implies (iii). It follows from Theorem 1.9 and 1.12 that $Q \subseteq' N$. Hence $Q \subseteq N \subseteq \hat{Q} = \hat{N}$ and $N = F_{\mathcal{L}}(\hat{Q},Q)$ by Theorem 1.9.

(iii) implies (i). By Theorem 1.16.

Proposition 1.20. Let $\mathcal{L}$ be a subclass of $\mathcal{M}$ satisfying (α), (σ) and Q be a module. Then Q is $\mathcal{L}$-injective if and only if every diagram (1) with N $\mathcal{L}$-dense in M can be made commutative.

Proof: If the condition is satisfied then $Q = r^1_{\mathcal{L}}(\hat{Q},Q)$ and Q is $\mathcal{L}$-injective by Theorem 1.3. Conversely, $M \subseteq N + r_{\mathcal{L}}(\hat{M},N)$ and it suffices to use Theorem 1.3 (iii) since $r_{\mathcal{L}}(\hat{M},N) \subseteq r_{\mathcal{L}}(\hat{M},N,f,Q)$ by Lemma 1.1.

§ 2. $(\mathcal{A},\mathcal{B})$-injective modules.

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be non-empty classes of modules. We say that a module Q is $(\mathcal{A},\mathcal{B})$-injective if every diagram (1) with $M/N \in \mathcal{A}$ and $M/\mathrm{Ker}\, f \in \mathcal{B}$ can be made commutative.

Baer's lemma 2.2. If $\mathcal{A}$ and $\mathcal{B}$ are abstract, hereditary and cohereditary classes of modules then a module Q is $(\mathcal{A},\mathcal{B})$-injective if and only if for every left ideal I of R with $R/I \in \mathcal{A}$ every homomorphism $f: I \longrightarrow Q$ with $R/\mathrm{Ker}\, f \in \mathcal{B}$ can be extended to $g: R \longrightarrow Q$.

Proof: We proceed to the sufficiency, the necessity being obvious. Suppose that there is a diagram (1) with

$M/_N \in \mathcal{A}$ and $M/_{Ker\ f} \in \mathcal{B}$ which cannot be made commutative. By Zorn's lemma, we can assume that f cannot be extended to any $N \subsetneqq K \subseteq M$. Let $b \in M \setminus N$ be arbitrary, $I = (N:b)$. Then $R/_I \cong (Rb + N)/_N$ lies in $\mathcal{A}$. Further, defining φ : $: I \longrightarrow Q$ by $\varphi(r) = f(rb)$ we have $Ker\ \varphi = (Ker\ f: b)$ and consequently $R/_{Ker\ \varphi} \cong (Rb + Ker\ f)/_{Ker\ f}$ lies in $\mathcal{B}$. Thus φ extends to $\psi : R \longrightarrow Q$ and hence f extends to g: $: \{N,b\} \longrightarrow Q$ given by $g(n + rb) = f(n) + \psi(r)$, a contradiction.

§ 3. Applications. Let $\mathcal{P}$ be a subclass of the class $\mathcal{R}$ of all couples (M,N), $N \subseteq M$. We say that $\mathcal{P}$ satisfies the condition (a) if $(M,N) \in \mathcal{P}$, $N \subseteq N' \subseteq M$ implies $(M,N') \in \mathcal{P}$.

Remark 3.1: Let $\mathcal{K}$, $\mathcal{P}$ be subclasses of $\mathcal{R}$ and $\mathcal{L} = \{\langle M,N,f,Q \rangle ;\ (M,N) \in \mathcal{P},\ f \in Hom_R(N,Q),\ (M, Ker\ f) \in \mathcal{K}\}$. Obviously, $\mathcal{L}$ satisfies (β) and (γ). Moreover, if both $\mathcal{K}$ and $\mathcal{P}$ satisfy (a) then $\mathcal{L}$ satisfies (α) and (δ).

Now we recall some basic definitions from the theory of preradicals (for details see [4] and [5]).

A preradical s for R-mod is any subfunctor of the identity functor, i.e. s assigns to each module M its submodule $s(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $s(M)$ into $s(N)$ by restriction. A preradical is said to be

- idempotent if $s(s(M)) = s(M)$ for every module M,
- a radical if $s\left(M/_{s(M)}\right) = 0$ for every module M,

- hereditary if $s(N) = N \cap s(M)$ for every submodule N of a module M.

A module M is s-torsion if $s(M) = M$ and s-torsionfree if $s(M) = 0$. If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in$ R-mod. The zero functor is denoted by zer and the identity functor by id.

For every $M \in$ R-mod we define $r_{\{M\}}(N) = \sum f(M)$, f ranging over all $f \in Hom_R(M,N)$. It is easy to see that $r_{\{M\}}$ is an idempotent preradical and, in fact, the smallest preradical for which M is a torsion module.

For a preradical s and modules $N \subseteq M$ let us define $C_s(N:M)$ by $C_s(N:M)/_N = s\left(M/_N\right)$. If $M = \hat{N}$ then we write simply $C_s(N) = C_s(N:\hat{N})$. Obviously, for $N_0 \subseteq N$, $M_0 \subseteq M$ and $f \in Hom_R(M,N)$ with $f(M_0) \subseteq N_0$ we have $f(C_s(M_0:M)) \subseteq$ $\subseteq C_s(N_0:N)$.

Definition 3.2. Let s and u be preradicals for R-mod. A submodule N of a module M is said to be s_u-dense in M if $M \leq C_s(N:C_u(M))$. A preradical s is said to be balanced if $A/_B \cong C/_D$ implies that B is s_{id}-dense in A if and only if D is s_{id}-dense in C.

Remark 3.3: The fact that N is s_{id}-dense in M means that N is s-dense in M in the sense of Beachy [1]. Further, N is s_{zer}-dense in M if and only if $M/_N$ is s-torsion and if s is hereditary then s_u-density means the same as s_{zer}-density for every preradical u.

Lemma 3.4. Let s and u be preradicals for R-mod and $N_1 \subseteq N_2 \subseteq N$ be modules. If N_1 is s_u-dense in N then N_2 is so.

Proof: Obvious since $N \subseteq C_s(N_1 : C_u(N)) \subseteq C_s(N_2 : C_u(N))$.

Definition 3.5. Let to every $M \in$ R-mod correspond four preradicals $s^{(M)}, t^{(M)}, u^{(M)}, v^{(M)}$. Let $\mathcal{P}$ be the class of all couples (M,N) of modules such that N is $s^{(M)}_{u^{(M)}}$-dense in M and $\mathcal{K}$ be the class of all couples (M,N) such that N is $t^{(M)}_{v^{(M)}}$-dense in M . Now let $\mathcal{L}$ be the class of all $\langle M,N,f,Q \rangle$ such that $(M,N) \in \mathcal{P}$, $f \in \mathrm{Hom}_R(N,Q)$ and $(M, \mathrm{Ker}\ f) \in \mathcal{K}$. We say that a module Q is (s,t,u,v)-injective if it is $\mathcal{L}$-injective.

Proposition 3.6. Every module Q has an (s,t,u,v)-injective envelope which is unique up to Q-isomorphism.

Proof: Both classes $\mathcal{P}$ and $\mathcal{K}$ satisfy Condition (a) by Lemma 3.4 so that it suffices to use Remark 3.1 and Theorem 1.12.

Lemma 3.7. Let s,t,u,v be preradicals for R-mod , A, B, M be modules, $A \subseteq B$ and $f \in \mathrm{Hom}_R(M,\hat{B})$ be such that $f^{-1}(A)$ is s_u-dense in M and $\mathrm{Ker}\ f$ is t_v-dense in M . Then $f(M) \subseteq C_s(A:\hat{B}) \cap t(\hat{B})$.

Proof: Easy.

Lemma 3.8. Let $A \subseteq B$ be modules and s,t,u,v preradicals for R-mod satisfying one of the following conditions:

(i) $u = t = id$, A is s_{id}-dense in B ,

(ii) s is idempotent and $u = zer$, $t = id$,

(iii) s is hereditary and $v = id$.

If, in the notation of 3.5, $s^{(M)} = s$, $t^{(M)} = t$,

$u^{(M)} = u$, $v^{(M)} = v$ for every $M \in R$-mod then $C_s(A:\hat{B}) \cap \cap\, t(\hat{B}) \subseteq r_{\mathcal{L}}(\hat{B}, A)$.

Proof: Put $M = C_s(A:\hat{B}) \cap t(\hat{B})$ and $N = A \cap t(\hat{B})$. It is easy to see that N is s_u-dense in M and 0 is t_v-dense in M from which the assertion follows easily.

Definition 3.9. Let s, t be preradicals for R-mod. We say that A is an (s,t)-dense submodule of B if $B \subseteq \subseteq A + (C_s(A:\hat{B}) \cap t(\hat{B}))$. An essential, (s,t)-dense submodule A of B is said to be (s,t)-essential in B.

Proposition 3.10. Let s,t be preradicals for R-mod and $A \subseteq B$ be modules. Then A is (s,t)-dense in B if and only if A is s_{id}-dense in B and $B = A + (B \cap t(\hat{B}))$.

Proof: Easy.

We say that the preradicals s, t, u, v for R-mod satisfy Condition $(*)$ if one of the following holds:

(i) $u = t = id$,

$(*)$ (ii) $u = zer$, $t = id$ and s is idempotent,

(iii) $v = id$ and s is hereditary.

Corollary 3.11: Under the notation of 3.5 let $s^{(M)} = s$, $t^{(M)} = t$, $u^{(M)} = u$, $v^{(M)} = v$ for every $M \in R$-mod. If s,t,u,v satisfy Condition $(*)$ and $A \subseteq B$ are modules then A is $\mathcal{L}$-dense in B if and only if A is (s,t)-dense in B.

Proof: The proof of the necessity is direct and the sufficiency follows immediately from 3.8.

Corollary 3.12: The following are equivalent for preradicals s, t, u, v for R-mod satisfying Condition $(*)$:

(i) Q is a direct summand in each extension N in which it is (s,t)-dense,

(ii) $Q \supseteq C_s(Q) \cap t(\hat{Q})$,

(iii) every diagram (1) with N (s,t)-dense in M can be made commutative,

(iv) Q is (s,t,u,v)-injective.

(See [1], 2.5.)

Proof: Conditions (iii) and (iv) are equivalent by 1.20 and 3.11. Further, by 3.11, Condition (i) means the same as that of Theorem 1.3. Now $r_{\mathcal{L}}(Q) = C_s(Q) \cap t(\hat{Q})$ by 3.7 and 3.8 and Theorem 1.3 finishes the proof since $\mathcal{L}$ satisfies Condition (α) by Lemma 3.4.

Corollary 3.13: Let s, t, u, v be preradicals for R-mod satisfying Condition (*). For any $Q \in$ R-mod define the sequence of modules Q_α inductively as follows: $Q_0 = Q$, $Q_{\alpha+1} = Q_\alpha + (C_s(Q_\alpha : \hat{Q}) \cap t(\hat{Q}))$ and $Q_\alpha = \bigcup_{\beta < \alpha} Q_\beta$, α limit. Then the module $\overline{Q} = Q_\alpha$ where $Q_\alpha = Q_{\alpha+1}$ is the smallest (s,t,u,v)-injective submodule of $\hat{Q}$ containing Q .

Proof: By Lemma 3.7, 3.8 and Theorem 1.9.

Lemma 3.14. Let t be a preradical, s a radical and $A \subseteq B \subseteq C$ be modules. If A is (s,t)-essential in B and B is (s,t)-essential in C then A is (s,t)-essential in C .

Proof: With respect to Proposition 3.10 it suffices to show that if A is s_{id}-dense in B and B is s_{id}-dense in C then A is s_{id}-dense in C . But this follows easily from the radical property of s .

Corollary 3.15: Let s, t, u, v be preradicals for R-mod satisfying Condition (*). If s is a radical and

$Q \in$ R-mod then $\bar{Q} = Q + (C_s(Q:\hat{Q}) \cap t(\hat{Q}))$ is the smallest (s,t,u,v)-injective submodule of $\hat{Q}$ containing Q . (See [1], 2.7 .)

Proof: In the notation of Corollary 3.13, Q is (s,t)-dense in Q_2 by Lemma 3.14, so that $Q_2 = Q_1$ and Corollary 3.13 completes the proof.

Corollary 3.16: Let s,t,u,v be preradicals for R-mod satisfying Condition (∗), $Q \in$ R-mod . The module Q is (s,t,u,v)-injective if and only if it has no proper (s,t)-essential extension.

Proof: By Corollary 3.11 and Theorem 1.14.

Corollary 3.17: Let s,t,u,v be preradicals for R-mod satisfying Condition (∗), s be a radical and Q, $N \in$ R-mod . The following are equivalent:

(i) N is a maximal (s,t)-essential extension of Q ,

(ii) N is an (s,t,u,v)-injective envelope of Q ,

(iii) N is an (s,t,u,v)-injective (s,t)-essential extension of Q .

Proof: It follows immediately from Lemmas 3.7, 3.8, 3.14 and Theorem 1.19.

Corollary 3.18 (Baer's lemma). Let r, s be hereditary preradicals for R-mod . Then a module Q is (s,t,zer,zer)-injective if and only if for every left ideal I s_{zer}-dense in R every homomorphism $f: I \longrightarrow Q$ with Ker f t_{zer}-dense in R can be extended to $g: R \longrightarrow Q$.

Proof: By Lemma 2.2.

Lemma 3.19. Let s be a hereditary preradical for R-mod and t be a balanced preradical. If $A \subseteq B$ are modules,

$f \in \mathrm{Hom}_R(M, C_s(A:\hat{B}) \cap t(\hat{B}))$, $g = if$ where i is the inclusion of $C_s(A:\hat{B}) \cap t(\hat{B})$ in B , then $g^{-1}(A)$ is s_{zer}-dense in M and $\mathrm{Ker}\, g$ is t_{id}-dense in M .

Proof: Easy.

Corollary 3.20: Let to every $M \in R$-mod correspond a hereditary preradical $s^{(M)}$, a balanced preradical $t^{(M)}$, $u^{(M)} = zer$, $v^{(M)} = id$ and let $\mathcal{L}$ be as in 3.5. If A , $B \in R$-mod , $A \subseteq B$ then $r_{\mathcal{L}}(\hat{B}, A) = \sum_M r_{\{M\}}\left(C_{s^{(M)}}(A:\hat{B}) \cap t^{(M)}(\hat{B})\right)$.

Proof: By 3.19.

Corollary 3.21: Under the hypotheses of Corollary 3.20 the following are equivalent for a module Q :

(i) Q is a direct summand in each extension N such that $N \subseteq Q + \sum_M r_{\{M\}}\left(C_{s^{(M)}}(Q:\hat{N}) \cap t^{(M)}(\hat{N})\right)$,

(ii) $Q \supseteq \sum_M r_{\{M\}}\left(C_{s^{(M)}}(Q) \cap t^{(M)}(\hat{Q})\right)$,

(iii) every diagram (1) with $M \subseteq N + \sum_U r_{\{U\}}\left(C_{s^{(U)}}(N:\hat{M}) \cap t^{(U)}(\hat{M})\right)$ can be made commutative,

(iv) Q is (s,t,zer,id)-injective.

Proof: By Corollary 3.20 and Theorem 1.3.

Corollary 3.22: Let $s^{(M)}$ and $t^{(M)}$ be as in 3.20. For any $Q \in R$-mod define the sequence of modules Q_α inductively as follows:

$Q_0 = Q$, $Q_{\alpha+1} = Q_\alpha + \sum_M r_{\{M\}}\left(C_{s^{(M)}}(Q_\alpha : \hat{Q}) \cap t^{(M)}(\hat{Q})\right)$ and $Q_\alpha = \bigcup_{\beta < \alpha} Q_\beta$, α limit. Then the module $\tilde{Q} = Q_\alpha$ where $Q_\alpha = Q_{\alpha+1}$ is the smallest (s,t,zer,id)-injective submodule of $\hat{Q}$ containing Q .

Proof: By Corollary 3.20 and Theorem 1.9.

Corollary 3.23: For every module M let $t^{(M)}$ be a balanced preradical for R-mod and $s^{(M)} = s$ be a hereditary radical. Then the module $\bar{Q} = Q + \sum_M r_{\{M\}}\left(C_s(Q:\hat{Q}) \cap \cap t^{(M)}(\hat{Q})\right)$ is the smallest (s,t,zer,id)-injective submodule of $\hat{Q}$ containing Q .

<u>Proof</u>: By Lemma 3.14 and Corollary 3.22.

References

[1] J.A. BEACHY: A generalization of injectivity, Pacif. J. Math. 41(1972),313-328.

[2] Bo STENSTRÖM: Rings and modules of quotients, Lecture Notes in Mathematics 237. Berlin-Heidelberg-New York:Springer 1971.

[3] L. BICAN: Preradicals and injectivity, Pacif. J. Math. 56(1975) (to appear).

[4] L. BICAN, P. JAMBOR, T. KEPKA, P. NĚMEC: Preradicals, Comment. Math. Univ. Carolinae 15(1974),75-83.

[5] L. BICAN, P. JAMBOR, T. KEPKA, P. NĚMEC: Hereditary and cohereditary preradicals (to appear).

[6] A.P. MIŠINA, L.A. SKORNJAKOV: Abelevy gruppy i moduli, Moskva 1969.

Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

(Oblatum 23.4. 1975)