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OPTIMAL AND SARD APPROXIMATION

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Abstract: The paper develops the previous Sard's idea of an approximation the error of which is minimal in the given space. It investigates the approximation in the space of complex-valued functions on which the Hilbert pseudonorm is given. The Sard approximation and an optimal approximation on some subspace is defined. The relation between the optimal and Sard approximation is studied. The theory is illustrated by the example of integration in the Sobolev space.

Key words: Optimal approximation, Hilbert pseudonorm, reproducing kernel, natural spline, quadrature, Sobolev space.

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Introduction. An approximation of linear functional A is understood as a linear combination $\sum_{i=1}^{n} \alpha_i g_i$ of "simpler" linear functionals $g_1, \dots, g_n$.

It is a natural requirement to define such an approximation the error of which would be minimal in the given space. A. Sard treated this idea particularly in the $C^m \langle a,b \rangle$ space in [4] and [5]. As A he took an integral $\int_a^b f\,dt$ and his approximating functionals were the Dirac measures. If the approximation is exact on all polynomials of degree $m - 1$ then the error has the form

$$\mathcal{R} f = \int_a^b \mathcal{K} f^{(m)} dt ,$$

where $\mathcal{K}$ is the so-called Peano kernel. Sard called this approximation the best approximation if $\int_a^b \mathcal{K}^2 dt$ is minimal. He generalized his results later in [6] dealing with an approximation of a linear operator from Banach space to Banach space.

The present paper develops the previous idea of Sard but in another direction than in [6]. An approximation is investigated in the space X of complex functions on which a Hilbert pseudonorm is given.

The paper is partially based on a preceding joint work [3]. Analogously to [3], the ambiguous subspace X_0, on which the pseudonorm is a norm, is constructed. Assuming that X_0 is a Hilbert space, we define the optimal approximation on it, i.e. the approximation which has the minimal error in the sense of its norm minimization. If X_0 is a space with reproducing kernel (for more details see [1], [7]) then we will facilitate the calculation of the optimal approximation. Further, an approximation that gives the minimal error with respect to the pseudonorm on X is defined. It is shown that this is a generalization of the previous definition of the best approximation given by Sard, and therefore we call it the Sard approximation.

The relation between the optimal and the Sard approximation is studied. An argument is given why the Sard approximation on X is preferred to the optimal one with respect to X_0 on X. However, it is considered as more convenient, in most cases, to use the optimal approximation on X_0. This approximation can be obtained from the Sard one by appropriately filling in the set of approximating functionals.

Finally, the theory is illustrated by an example of integration in the Sobolev space. The theory can also be used for determining optimal differential schemes.

1. General theory

1.1. Notation and assumptions. Let X be a space of complex valued functions which are defined on the set Q. Let $\| \ \|$ denote a Hilbert pseudonorm on X, i.e. the map $X \longrightarrow \mathbb{R}_1$ satisfying

$$\|x + y\| \leq \|x\| + \|y\| \qquad x, y \in X$$

(i) $\|\alpha x\| = |\alpha| \cdot \|x\| \qquad \alpha \in C$ (C denotes the set of complex numbers).

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

Let the kernel of the pseudonorm $M = \{x \in X, \|x\| = 0\}$ satisfy

(ii) $\dim M < \infty$.

Further let $\mathfrak{M}$, $\mathfrak{M} \subset X^*$ be such set that

(iii) $M \cap \mathfrak{M}^{\perp} = \{0\}$ x) .

We are given linear functionals A and $g_1, \dots, g_n$ defined on X and continuous in a certain algebraic direct complement of the space M in X. We shall denote it generally by X_M. We suppose that $g_1, \dots, g_n$ are linearly independent on X_M and

(iv) $M \cap G^{\perp} = \{0\}$ where $G = (g_1, \dots, g_n)$.

We shall consider approximations of A in the form

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x) $\mathfrak{M}^{\perp}$ denotes the annihilator of $\mathfrak{M}$.

$\sum_{i=1}^{m} \alpha_i g_i$ and by R^{α} denote the remainder of approximation, i.e. $R^{\alpha} = A - \sum_{i=1}^{n} \alpha_i g_i$, where $\alpha = (\alpha_1, \dots, \alpha_n)$. By R_M we denote an arbitrary remainder of approximation that vanishes on M ; let α_M be the coefficients of this approximation.

1.2. Optimal approximation. Similarly as in [3], we use

Lemma 1.1. Under the assumptions (i),(ii),(iii) there exists an m-tuple of linear functionals $L = (\ell_1, \dots, \ell_m)$, $\ell_i \in \mathcal{M}$, $i = 1, \dots, m$ such that the space $X_0 = \{x \in X ,\ L(x) = 0\}$ is an algebraic direct complement of M , i.e. $X = X_0 \oplus M$.

Remark. We shall call such a space X_0 the DC-space.

According to Lemma 1.1 it follows that the pseudonorm is a norm on X_0 . So we can prove the following proposition.

Proposition 1.1. If F is a continuous functional on some DC-subspace then F is continuous on all DC-subspaces.

Hence the continuity of the functionals $A, g_1, \dots, g_n$ does not depend on the choice of a DC-subspace. Suppose

(v) X_0 is complete with respect to the norm $\| \ \|$ (i.e. X_0 is a Hilbert space).

Remark. It is easy to prove that the validity of (v) implies the validity of (v) for all complements X_M of M in X .

Definition 1.1. A functional $\sum_{i=1}^{m} \alpha^0_i g_0$ is called an

<u>optimal approximation</u> on X_o with respect to G if there exists an n-tuple of coefficients α^o such that the remainder of the approximation $R^o \equiv R^{\alpha^o}$ satisfies

$$\sup_{\substack{\|f\| \leq 1 \\ f \in X_o}} | R^o f | \leq \sup_{\substack{\|f\| \leq 1 \\ f \in X_o}} | R^{\alpha} f | \qquad \forall \alpha \in \mathbb{R}_m$$

The Riesz representation theorem guarantees the existence and uniqueness of functions $\varphi, \xi_1, \ldots, \xi_n$ from the space X_o which are the representatives of $A, g_1, \ldots, g_n$. Let us denote $Z = \text{span} \{\xi_i,\ i = 1, \ldots, n\}$ the subspace of the space X_o. Because of the linear independence of $g_1, \ldots$ $\ldots, g_n$ we have that $\dim Z = n$. It is easy to prove the following theorems.

<u>Theorem 1.1.</u> Under the assumptions (i) - (v) there exists an optimal approximation with respect to G on X_o. Moreover, the optimal approximation is unique iff $g_1, \ldots, g_n$ are linearly independent on X_o.

<u>Theorem 1.2.</u> Let (i) - (v) hold. Then the approximation on X_o with respect to G is optimal iff it is exact for all functions belonging to Z.

If X_o satisfies another assumption

(vi) X_o is a space with a reproducing kernel, the so-called RK-space (see [3])

then the optimal approximation on X_o can be well evaluated in most cases. The fulfilling of the assumption (vi) does not depend on the choice of the m-tuple of functionals L, i.e. if any DC-space is RK-space then all DC-spaces are DKspaces. This is easy to prove similarly as in [3] by means of Proposition 1.1.

If K_t denotes the reproducing kernel in X_0 then we have

$$g_i(K_t) = (K_t, \xi_i) = \overline{(\xi_i, K_t)} = \overline{\xi_i(t)}$$

$$A(K_t) = (K_t, \varphi) = \overline{(\varphi, K_t)} = \overline{\varphi(t)} \quad .$$

<u>Remark</u>. In the case that the functionals $g_1, \dots, g_n$ reproduce the values of the function, i.e. $g_i: f \longrightarrow f(t_i)$, $i = 1, \dots, n$, then $Z = \operatorname{span}\{K_{t_1}, \dots, K_{t_n}\}$ (see the Application).

Further we shall assume only (v).

1.3. <u>Sard approximation</u>. It follows from the assumption (iv) that there exists at least one approximation with respect to G which is exact on M. Namely, there exist m functionals $g_{k_1}, \dots, g_{k_m}$ from G, so that the matrix $(g_{k_i}(f_j))_{i,j=1}^m$ is regular, where $f_1, \dots, f_m$ is a basis of M. Hence there exists $\alpha_M = (\alpha_1, \dots, \alpha_n)$ such that

$$(1.1) \qquad \sum_{i=1}^{n} \alpha_i g_i(f_j) = Af_j \ , \quad j = 1, \dots, m \quad .$$

<u>Definition 1.2</u>. We call a functional $\sum_{i=1}^{n} \alpha_i^S g_i$ the <u>Sard approximation</u> with respect to G, if it is exact on M (i.e. $\forall f \in M \quad R^S f = R^{\alpha^S} f = 0$) and, moreover,

$$(1.2) \qquad \sup_{\substack{\|f\| \le 1 \\ f \in X}} |R^S f| \le \sup_{\substack{\|f\| \le 1 \\ f \in X}} |R_M f|$$

holds for arbitrary R_M .

<u>Remark</u>. It is easy to verify that in the definition 1.2 the right hand side term of (1.2) can be replaced by the expression:

$$\sup_{\substack{\|f\| \leq 1 \\ f \in X}} |R^{\alpha} f| \qquad \forall \alpha \in \mathbb{R}_m$$

<u>Theorem 1.3</u>. Assuming (i) - (iv) to hold there exists a unique Sard approximation with respect to G .

<u>Proof</u>. a) Existence. Let us define the function

$$\varphi : \alpha_M \longrightarrow \sup_{\substack{\|f\| \leq 1 \\ f \in X}} |Af - \sum_{i=1}^{m} \alpha_i g_i(f)| \ , \text{ where } \alpha_M = (\alpha_1, \dots, \alpha_n) \ .$$

We shall prove that this function is continuous on its domain, i.e. on each vector of coefficients from $\mathbb{R}_n$ which is the solution of m equations (1.1). This domain is a closed set. Further,

$$|\varphi(\alpha_M) - \varphi(\bar{\alpha}_M)| \leq \sup_{\substack{\|f\| \leq 1 \\ f \in X_M}} \Big| |Af - \sum_{i=1}^{m} \alpha_i g_i(f)| -$$

$$- |Af - \sum_{i=1}^{m} \bar{\alpha}_i g_i(f)| \Big| \leq \sup_{\substack{\|f\| \leq 1 \\ f \in X_M}} |\sum_{i=1}^{m} (\alpha_i - \bar{\alpha}_i) g_i(f)| \leq$$

$$\leq \max_i \sup_{\substack{\|f\| \leq 1 \\ f \in X_M}} |g_i(f)| \sum_{j=1}^{m} |\alpha_j - \bar{\alpha}_j| \ ,$$

where $\bar{\alpha}_M = (\bar{\alpha}_1, \dots, \bar{\alpha}_n)$.

Since the functionals $g_1, \dots, g_n$ are continuous on X_M it evident that the function φ is continuous in its domain, for example with the norm $\|\alpha\| = \max_i |\alpha_i|$. If we prove that

$$(1.3) \qquad \varphi(\alpha_M) \longrightarrow \infty \quad \text{if} \quad \|\alpha_M\| \longrightarrow \infty$$

then the function φ will attain its minimum. The functionals $g_1,\dots,g_n$ are linearly independent on a certain subspace X_M, i.e. on an algebraic complement of the space M. Thus there exist functions $\hat{f}_j \in X \setminus M$, $\hat{f}_j \neq 0$, $j = 1,\dots$ $\dots,n$ such that the matrix $(g_i(\hat{f}_j))_{i,j=1}^n$ is regular. We have

$$\varphi(\alpha_M) = \sup_{\substack{f \neq 0 \\ f \in X}} \frac{|Af - \sum_1^n \alpha_i g_i(f)|}{\|f\|} \geq \frac{\big|\,|A\hat{f}_j| - |\sum_1^n \alpha_i g_i(\hat{f}_j))\big|}{\|f_j\|}$$

$$\forall j = 1,\dots,n .$$

Letting $\|\alpha_M\| \longrightarrow \infty$ we see that $\max_j |\sum_1^n \alpha_i g_i(f_j)| \longrightarrow$ $\longrightarrow \infty$ and hence (1.3) holds.

b) Uniqueness. If α^S denotes the n-tuple of coefficients of a Sard approximation then

$$\|R^S\|_{X_M} = \sup_{\substack{\|f\| \leq 1 \\ f \in X}} |R^S f| \leq \sup_{\substack{\|f\| \leq 1 \\ f \in X_M}} |R_M f| = \|R_M\|_{X_M}$$

for any n-tuple α_M satisfying (1.1). Since X_M is a space with Hilbert norm, $g_1,\dots,g_n$ are linearly independent on X_M and because of the fact that the domain of the coefficients α_M is a closed set, we get the uniqueness of coefficients α^S. This follows from the orthogonal projection theorem (on a convex closed set in Hilbert space). The proof would be similar to that of Theorem 1.1.

1.4. <u>The relation between optimal and Sard approximations</u>. Let us observe that the Sard approximation with respect to G has been defined independently of the DC-subs space. Provided that (iv) holds, one can assume without loss of generality that the matrix $(g_i(f_j))_{i,j=1}^n$ is regular, where $f_1,\dots,f_m$ is a basis of M.

If we put

$$Y_0 = \{x \in X,\ g_i(x) = 0\ ,\quad i = 1,\dots,m\}$$

then Y_0 is a DC-space for $\mathfrak{M} = \{g_i,\ i = 1,\dots,m\}$ and a Hilbert space, too.

<u>Theorem 1.4</u>. Let us suppose that (i) - (v) hold and let $g_{m+1},\dots,g_n$ be linearly independent on Y_0. If α_i^S, $i = 1,\dots,n$ are the coefficients of the Sard approximation with respect to G on X and α_i^o, $i = m+1,\dots,n$ the coefficients of the optimal approximation on Y_0 with respect to $g_{m+1},\dots,g_n$ on Y_0, then

$$\alpha_i^S = \alpha_i^o\ ,\quad i = m+1,\dots,n\ .$$

<u>Proof</u>. Let us define an approximation with respect to G with coefficients $\alpha^\vee = (\alpha_1^\vee,\dots,\alpha_n^\vee)$ such that $\alpha_i^\vee = \alpha_i^o$, $i = m+1,\dots,n$ and $\alpha_i^\vee$, $i = 1,\dots,m$ is the solution of m linear algebraic equations

$$\sum_{i=1}^{m} \alpha_i^\vee g_i(f_k) = Af_k - \sum_{i=m+1}^{m} \alpha_i^o g_i(f_k) \qquad k = 1,\dots,m$$

The remainder $R^\vee \equiv R^{\alpha^\vee}$ of the approximation defined in this way satisfies the requirements of the Sard approximation. Indeed,

$$\check{R}f = 0 \quad \forall f \in M ,$$

$$\sup_{\substack{\|f\| \le 1 \\ f \in X}} | R^S f | = \inf_{\hat{\alpha}_M} \sup_{\substack{\|f\| \le 1 \\ f \in Y_o}} | R^{\hat{\alpha}_M} f | \ge \sup_{\substack{\|f\| \le 1 \\ f \in Y_o}} | R^o f | = \sup_{\substack{\|f\| \le 1 \\ f \in Y_o}} | \check{R} f | =$$

$$= \sup_{\substack{\|f\| \le 1 \\ f \in X}} | \check{R} f |$$

where $\hat{\alpha}_M$ denote $(\alpha_{m+1}, \dots, \alpha_n)$ for any $\alpha_M = (\alpha_1, \dots$ $\dots, \alpha_n)$. The converse inequality is trivial. Due to the uniqueness of the Sard approximation, the assertion of the theorem follows immediately.

Remark. The Sard approximation with respect to G is exact not only on M but also on the functions g_{m+i} , $i =$ $= 1, \dots, n - m$. It should be observed that the considered expression is an approximation with respect to G . This means that it cannot be an optimal approximation on the space Y_o with respect to G .

It seems natural to ask whether it would be better to use some of the optimal approximations on X with respect to G than the Sard approximation with respect to G . To compare both these approximations, the following theorem is of importance.

Let X_o , Z denote the spaces defined in Section 1.2 and $g_1, \dots, g_n$ be linearly independent functionals on M , which satisfy (v).

Due to the assumption (iv) and since the optimal approximation is exact on all functions from Z we have

Theorem 1.5. Let us assume that (i) - (v) hold. The optimal approximation on X_0 is not exact on M iff the following implication holds:

$$(1.4) \qquad \exists\, p \in M\,,\ \gamma \in Z:\ G(p) = G(\gamma) \Longrightarrow Ap \neq A\gamma\,.$$

Theorem 1.6. Assuming that (i) - (iv) hold, we have

$$(1.5) \qquad \sup_{\substack{\|f\| \leq 1 \\ f \in X_0}} |R^o f| < \sup_{\substack{\|f\| \leq 1 \\ f \in X_0}} |R^S f|$$

for any DC-subspaces X_0 which satisfies (v) and (1.4).

Proof. If a DC-subspace X_0 exists such that it satisfies (v), (1.4) and such that in (1.5) the equality holds then $\alpha^o = \alpha^S$ would be valid because of the uniqueness of the optimal approximation on X_0. But this is a contradiction with Theorem 1.5.

Corollary. If the task is to find the value of the functional A for a function $f \in X$, we must consider whether f lies in some DC-subspace X_0 fulfilling (1.4) and whether the given $g_1, \dots, g_n$ are linearly independent on X_0. Thus, it is better to use the optimal approximation on X_0 than the Sard approximation on X. When the answer is negative, we shall use the Sard approximation because

$$\sup_{\substack{\|f\| \leq 1 \\ f \in X}} |R^S f| < \sup_{\substack{\|f\| \leq 1 \\ f \in X}} |R^o f| = \infty$$

In this case we shall say that the Sard approximation is "universal" with respect to all DC-subspaces satisfying (v) and (1.4).

2. Application

Let us consider the Sobolev space $X = W_2^m \langle a,b \rangle$, $m \in \mathbb{N}$ with the given Hilber pseudonorm $\| f \|^2 = \int_a^b | f^{(m)}(t) |^2 \, dt$. Then M is π_{m-1}, i.e. the set of polynomials of degree less than m. If $\mathfrak{M}$ is the set of all functionals $F: f \longrightarrow f(t)$, $t \in Q_0 \subset \langle a,b \rangle$, where card $Q_0 \geq m$, then (iii) holds. If we choose m different points $x_1, \dots, x_m$ from Q_0 and put $X_0 = \{ f \in X,\ f(x_k) = 0,\ x_k \in Q_0,\ k = 1, \dots, m \}$ then X_0 is a DC-space satisfying (v). Due to the continuous imbedding of $W_2^m \langle a,b \rangle$ with usual norm into the space $C \langle a,b \rangle$, the condition (vi) is satisfied. The reproducing kernel of X_0 is the vector-function $K_t \in X_0$ such that $K_t \in N_{2m-1}(x_1, \dots, x_m, t)$, i.e. K_t is a natural spline of degree $2m - 1$ with knots $x_1, \dots, x_m, t$, see [3].

Let A be a functional given on X by $Af = \int_a^b f \, dt$ and the functionals g_i by $g_i(f) = f(t_i)$, $t_i \in \langle a,b \rangle$, $i = 1, \dots, n$, $\{x_i\}_1^m \cap \{t_i\}_1^n = \emptyset$. Under the assumption $n \geq m$ (iv) holds. The $g_1, \dots, g_n$ are continuous and linearly independent on X_0.

In this case we shall call the optimal, resp. the Sard approximation, the optimal, resp. the Sard quadrature.

We shall now demonstrate that the Sard quadrature coincides with the best quadrature which is given by Sard in [4].

Definition 2.1. (Sard). Let $t_1,\dots,t_n$ be different points from $\langle a,b\rangle$ and m an integer such that $m \leq n$. The remainder of a quadrature exact on π_{m-1}

$$\int_a^b f\,dt \sim \sum_{k=1}^{n} a_k f(t_k)\ , \quad \text{where} \quad f \in C^m\langle a,b\rangle$$

can be expressed in the form

$$(2.1) \qquad R f = \int_a^b \mathcal{K}(t) f^{(m)}(t)\,dt,$$

where $\mathcal{K}$ is the so-called Peano kernel (see [5]). The quadrature exact on π_{m-1} with the remainder R^* is called the best quadrature if the corresponding Peano kernel

(2.2) $\mathcal{K}^*$ has the minimal L_2-norm among all Peano kernels of quadratures exact on π_{m-1}.

The density of $C^m\langle a,b\rangle$ in $W_2^m\langle a,b\rangle$ guarantees that (2.1) holds for all $f \in X$. But (2.1) is equivalent to the requirement

$$\sup_{\|f^{(m)}\|_{L_2} \leq 1} |R^* f| = \inf_{\alpha_M} \sup_{\|f^{(m)}\|_{L_2} 1} |R_M f| \qquad f \in X$$

We introduce a linear map $J: W_2^m\langle a,b\rangle \longmapsto L_2\langle a,b\rangle$ such that $Jf = f^{(m)}$ and a functional

$$\Phi(g) = \int_a^b g \cdot \mathcal{K}^*\,dt\ , \quad g \in L_2\langle a,b\rangle\ ,$$

then we have

$$\sup_{\substack{\|f^{(m)}\|_{L_2} \leq 1 \\ f \in X}} |R^* f| = \sup_{\substack{\|f^{(m)}\|_{L_2} \leq 1 \\ f \in X}} |\Phi(Jf)| = \sup_{\|g\|_{L_2} \leq 1} |\Phi(g)| =$$

$$= \|\Phi\|_{L_2} = \|\mathcal{K}^*\|_{L_2}$$

Therefore both quadrature formulae are equivalent.

If we now apply our theory on this special case, we shall receive the following results. Let D_n denote the sequence of points $t_1,\dots,t_n$.

<u>Theorem 2.1.</u> There exists the unique optimal quadrature on the space X_o with respect to the D_n .

<u>Theorem 2.2.</u> The quadrature is optimal in X_o iff it is exact on the space $Z = \{ f \in N_{2m-1}(x_1,\dots,x_m,t_1,\dots,t_n)$, $f(x_i) = 0$, $i = 1,\dots, m \}$.

<u>Theorem 2.3.</u> There exists the unique Sard quadrature with respect to D_n .

Let us take the first m points $t_1,\dots,t_m$ from D_n and put $k = n - m$, $Y_o = \{ f \in X$, $f(t_i) = 0$, $i = 1,\dots,m \}$. The other k functionals $g_i: f \longrightarrow f(t_i)$, $i = m + 1,\dots$ $\dots, n$ are linearly independent on RK and DC-space Y_o . Let us denote $D_k = \{ t_{m+1} \dots t_n \}$.

<u>Theorem 2.4.</u> If α_i^S , $i = 1,\dots, n$ are the coefficients of the Sard quadrature with respect to D_n , and α_i^o , $i = m + 1,\dots,n$ are the coefficients of the optimal quadrature in Y_o with respect to D_k then $\alpha_i^S = \alpha_i^o$, $i = m + 1,\dots,n$.

<u>Corollary.</u> The Sard quadrature is exact on $M \oplus Z =$ $= \pi_{m-1} \oplus \{ f \in N_{2m-1}(t_1,\dots,t_n)$, $f(t_i) = 0$, $i = 1,\dots, n \} =$ $= N_{2m-1}(t_1,\dots,t_n)$, which is a known result (see [2]). It is possible to obtain the coefficients of the Sard quadrature by solving $n + m$ linear independent equations:

$$\sum_{1}^{n} \alpha_i^S t_i^r = \int_a^b t^r \, dt \ , \quad r = 0,\ldots,m-1$$

$$\sum_{i=1}^{n} \alpha_i^S (t_i - t_j)_+^{2m-1} + g(t_j) = \int_a^b (t - t_j)_+^{2m-1} \, dt \ ,$$

$$j = 1,\ldots,n \ , \quad t_+ = \begin{cases} t \ \ldots \ t > 0 \\ 0 \ \ldots \ t \le 0 \end{cases}$$

for α_i^S , $i = 1,\ldots,n$ and m coefficients of $g \in \pi_{m-1}$.

To obtain the optimal quadrature with respect to D_k on Y_0 it is sufficient to evaluate the Sard quadrature with respect to D_n .

Lemma 2.1. There exists a function $p \in \pi_{m-1}$ such that if $\gamma \in Z$ satisfies $G(\gamma) = G(p)$ then

$$\int_a^b p \, dt \neq \int_a^b \gamma \, dt \ .$$

Proof. We shall find the function $q_0 \in N_{2m-1}(x_1,\ldots,x_m,t_1,\ldots,t_n)$ such that $q_0(t_i) = 0$, $i = 1,\ldots,n$ and the values of which in $x_1,\ldots,x_n$ will be chosen to satisfy $\int_a^b q_0 \, dt \neq 0$. There exists a unique function q_0 given by values $q_0(t_1),\ldots,q_0(t_n)$ and $q_0(x_1),\ldots,q_0(x_m)$ (see [2]). Using the values $q_0(x_1),\ldots,q_0(x_m)$ at $x_1,\ldots,x_m$ we construct an interpolation polynomial $p_0 \in \pi_{m-1}$. If we define the function $\gamma_0 = p_0 - q_0$, then $\gamma_0 \in Z$ because of $\gamma_0(x_i) = p_0(x_i) - q_0(x_i) = 0$, $i = 1,\ldots,n$ and $\gamma \in N_{2m-1}(x_1,\ldots,x_m,t_1,\ldots,t_n)$.
However, $\int_a^b \gamma_0 \, dt \neq \int_a^b p_0 \, dt$ as required.

Due to validity of (1.4) we get

Theorem 2.5. There is no RK and DC-space X_0 such that the optimal quadrature is exact on π_{m-1} .

Corollary. The Sard quadrature is "universal" with respect to all RK and DC-subspaces.

R e f e r e n c e s

[1] N. ARONSZAJN: Theory of reproducing kernels, Trans. Amer. Math. Soc. 68(1950),337-404.

[2] T.N.E. GREVILLE: Introduction to spline functions, in Theory and applications of spline functions, Acad. Press, New York,(1969),1-35,Greville ed.

[3] J. MILOTA, C. STRÁNSKÁ: Interpolation and reproducing kernels, to appear.

[4] A. SARD: Best approximate integration formulas, best approximation formulas, Amer. J. Math. 71(1949), 80-91.

[5] A. SARD: Linear Approximation, Math. Surv. No 9, Amer. Math. Soc., Providence (1963).

[6] A. SARD: Optimal Approximation, J. Functional Anal. 1 (1967),222-244.

[7] H.S. Shapiro: Topics in approximation theory, Lecture Notes No. 181, Springer-Verlag 1971.

[8] I.J. SCHOENBERG: On monospline of least deviation and best quadrature formulae, J. SIAM Numer. Anal. Ser. B2(1965),144-170 and J. SIAM Numer. Anal. 3(1966),321-328.

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