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CONVERGENCE OF A DUAL FINITE ELEMENT METHOD IN R_n

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Abstract: Using the dual variational formulation of the elliptic second order problems, the question arises to construct suitable subspaces of admissible vector-funcions in R_n. In the paper a possible system of piecewise linear functions is shown and the rate of convergence $O(h^2)$ proved, provided the exact solution is sufficiently regular.

Key words: Finite elements, dual variational formulation, equilibrium models.

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1. Introduction. Let $\Omega \subset R_n$ be a bounded domain with Lipschitz boundary (cf. [2]), $k \geq 0$ integer. By $W^{k,2}(\Omega)$ we denote the set of real functions, which are square-integrable together with their generalized derivatives up to the order k, $W^{0,2}(\Omega) = L_2(\Omega)$, $[W^{k,2}(\Omega)]^m = \underbrace{W^{k,2}(\Omega) \times \dots \times W^{k,2}(\Omega)}_{m\text{-times}}$ with the norm

$$\| v \|_{k,\Omega} = \Big(\sum_{i=1}^{m} \| v_i \|^2_{k,\Omega}\Big)^{1/2}, \quad v = (v_1, \dots, v_m),$$

where

$$\| v_i \|_{k,\Omega} = \Big(\int_\Omega \sum_{|\alpha| \leq k} | D^\alpha v_i(x)|^2 \, dx\Big)^{1/2}.$$

Let $C^k(\bar{\Omega})$ denote the space of continuous functions, the derivatives of which up to the order k are also continuous and continuously extendible onto $\bar{\Omega}$,

$$[C^k(\bar{\Omega})]^m = \underbrace{C^k(\bar{\Omega}) \times \ldots \times C^k(\bar{\Omega})}_{\text{m-times}} ,$$

with the norm

$$\| v \|_{[C^k(\bar{\Omega})]^m} = \max_{i=1,\ldots,m} \left(\max_{|\alpha| \leq k} \left(\max_{x \in \bar{\Omega}} | D^\alpha v_i(x) | \right)\right) .$$

Let $M \subset R_n$. We denote by $P_k(M)$ the space of all polynomials in n-variables of the order at most k with the domain M .

Let us consider the differential operator

$$\mathcal{A} u = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j}\right) ,$$

satisfying the following conditions:

(1.1) $a_{ij} \in L_\infty(\Omega)$, $a_{ij}(x) = a_{ji}(x) \quad \forall\, i,j$, $\forall x \in \Omega$

(1.2) $\exists\, \alpha = \text{const.} > 0$,

$$\sum_{i,j=1} a_{ij}(x)\, \xi_i \xi_j \geq \alpha \| \xi \|^2 \quad \forall \xi \in R_n$$

almost everywhere in Ω .

Let the boundary Γ consist of three disjoint parts Γ_u , Γ_g , $\mathcal{R}$ such that Γ_u is open in Γ , $\Gamma_u \neq \emptyset$, $\text{mes}_{n-1} \mathcal{R} = 0$, Γ_g either empty or open in Γ and

$$\Gamma = \Gamma_u \cup \Gamma_g \cup \mathcal{R} .$$

We shall solve the following problem:

$$(1.3) \quad \left| \begin{array}{ll} \mathcal{A} u = f & \text{in } \Omega , \\ u = \bar{u} & \text{on } \Gamma_u , \\ \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} n_i = g & \text{on } \Gamma_g , \end{array} \right.$$

where $f \in L_2(\Omega)$, $\bar{u} \in W^{1,2}(\Omega)$, $g \in L_2(\Gamma_g)$ are assigned,

n_i are components of the unit outward normal to Γ.

We set:

$$V = \{ v \mid v \in W^{1,2}(\Omega), \quad v = 0 \text{ on } \Gamma_u \}.$$

A function $u \in W^{1,2}(\Omega)$ will be called weak solution of the problem (1.3), if

$$u \in \bar{u} + V,$$

$$\mathcal{L}(u) = \min_{v \in \bar{u}+V} \mathcal{L}(v),$$

where

$$\mathcal{L}(v) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx - \int_\Omega f v \, dx - \int_{\Gamma_g} g v \, d\Gamma.$$

Denoting

$$H = [L_2(\Omega)]^n,$$

$$\|\lambda\|^2 = \sum_{i=1}^n \int_\Omega \lambda_i^2 dx, \quad \lambda = (\lambda_1, \dots, \lambda_n),$$

we define the bilinear form

$$(\lambda, \mu)_H = \sum_{i,j=1}^n \int_\Omega b_{ij} \lambda_i \mu_j \, dx, \qquad \forall \lambda, \mu \in H,$$

where b_{ij} are the entries of the matrix $[a^{-1}]$ inverse to $[a]$.

Obviously, positive constants c_1, c_2 exist such that

$$c_1 \|\lambda\| < \|\lambda\|_H < c_2 \|\lambda\|,$$

where

$$\|\lambda\|_H^2 = (\lambda, \lambda)_H.$$

Moreover, we introduce

$$\lambda_i(v) = \sum_{j=1}^{n} a_{ij} \frac{\partial v}{\partial x_j} \quad , \quad \lambda(v) = (\lambda_1(v), \ldots, \lambda_n(v)),$$

$$\Lambda_{f,g} = \{\lambda \mid \lambda \in H,\ B(\lambda, v) = \int_\Omega f\, v\, dx + \int_{\Gamma_g} g\, v\, d\Gamma,\ \forall\, v \in V\}$$

where

$$B(\lambda, v) = \sum_{i=1}^{n} \int_\Omega \lambda_i \frac{\partial v}{\partial x_i}\, dx\ .$$

<u>Theorem 1.1</u> (The minimum of complementary energy). Let u be a weak solution of the problem (1.3). Then the functional

$$\mathcal{S}(\lambda) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} b_{ij}\, \lambda_i\, \lambda_j\, dx - B(\lambda, \bar{u})$$

attains its minimum on the set $\Lambda_{f,g}$, if and only if $\lambda = \lambda(u)$.

For the <u>proof</u> we refer to [1].

We have a variational problem of a minimum of a quadratic functional on a closed convex set $\Lambda_{fg} \subset H$. As usual, the problem can be interved to a similar one, but on a linear space subspace $\Lambda_{0,0} \equiv H_2 \subset H$. In fact, we may write

$$\Lambda_{f,g} = \bar{\lambda} + \Lambda_{0,0}\ ,$$

where $\bar{\lambda}$ is any fixed element of $\Lambda_{f,g}$.

It is easy to show that the equivalent problem is to find $\chi^0 \in H_2$ such that

(1.4) $$\Phi(\chi^0) = \min_{\chi \in H_2} \Phi(\chi)$$

where

$$\Phi(\chi) = \frac{1}{2}(\chi, \chi)_H - F(\chi) ,$$

$$F(\chi) = - \int_\Omega \sum_{i,j=1}^{n} b_{ij} \bar{\lambda}_j \chi_i \, dx - B(\chi, \bar{u}) .$$

Let $h \in (0,1\rangle$ and let $\{V_h\}$ be a system of finite-dimensional subspaces of H_2. We define the following procedure:

$$(1.5) \qquad \text{to find } \chi_h^0 \in V_h \text{ such that} \quad \Phi(\chi_h^0) = \min_{\chi \in V_h} \Phi(\chi) .$$

Theorem 1.2. To every $h \in (0,1\rangle$ there exists precisely one $\chi_h^0 \in V_h$ satisfying (1.5) and it holds

$$(1.6) \quad \|\chi^0 - \chi_h^0\|_H \leq \inf_{\chi \in V_h} \|\chi^0 - \chi\|_H \leq C_2 \inf_{\chi \in V_h} \|\chi^0 - \chi\| .$$

The proof can be found in [1] .

2. Construction of subspaces V_h . Let $\Sigma = \{a_i\}_{i=1}^{n+1}$ be the set of $(n + 1)$ points in R_n , $(n \geq 2)$ such that their coordinates $a_i = (a_{1i}, \dots, a_{ni})$ form a regular matrix

$$A = \begin{bmatrix} a_{11}, \dots & a_{1i}, \dots & a_{1n+1} \\ a_{21}, \dots & a_{2i}, \dots & a_{2n+1} \\ \vdots & \vdots & \vdots \\ a_{n1}, \dots & a_{ni}, \dots & a_{nn+1} \\ 1, \dots & 1, \dots & 1 \end{bmatrix}$$

The closed convex hull of Σ will be called n-simplex K in R_n , $a_i \in \Sigma$ its vertices and we write

$$K = \overline{\operatorname{conv} \Sigma} .$$

The assumption on the matrix A yields that the system

$$x_i = \sum_{j=1}^{n+1} a_{ij} \lambda_j(x) ,$$

$$1 = \sum_{j=1}^{n+1} \lambda_j(x)$$

has a unique solution $\lambda(x) = (\lambda_1(x), \ldots, \lambda_{n+1}(x))$ for any point $x \in R_n$. The components $\lambda_i(x)$ will be called <u>barycentric coordinates</u> of the point x with respect to the vertices $a_1, \ldots, a_{n+1}$. Thus the n-simplex K can be characterized by means of the barycentric coordinates:

$$x \in K \Longleftrightarrow 0 \leqq \lambda_i(x) \leqq 1 \qquad i = 1, \ldots, n+1$$
$$\sum_{i=1}^{n+1} \lambda_i(x) = 1 .$$

By $(n-1)$-dimensional <u>side</u> of K we call the closed convex hull of an arbitrary n-tuple of points of Σ. Consequently, the total number of $(n-1)$-dimensional sides of K equals

$$\binom{n+1}{n} = n + 1 .$$

Each vertex a_i belongs to n sides of K.

It is well known that the set Σ is $P_1(K)$-unisolvent, i.e. for any $\alpha_1, \ldots, \alpha_{n+1} \in R_1$ there exists precisely one polynomial $p \in P_1(K)$ such that $p(a_i) = \alpha_i$, $i = 1, \ldots, n+1$.

Let $\Sigma^i = \{a_{i_j}\}_{j=1}^{n}$, where $a_{i_j} \in \Sigma$. Then $S_i = \overline{\operatorname{conv} \Sigma^i}$ is a $(n-1)$-dimensional side of K.

We define a mapping $T_i \in \mathcal{L}([W^{1,2}(K)]^m, L_2(S_i))$ by the relation

$$T_i v = v|_{S_i} \cdot n^{(i)} = \sum_{j=1}^{n} v_j n_j^{(i)} ,$$

where $v|_{S_i}$ denotes the trace of v on S_i and $n^{(i)}$ is the unit outward normal to S_i .

Lemma 2.1 Let $\gamma_j^{(i)}$ be $(n+1)n$ real numbers ($i = 1,\dots,n+1$ and $j = 1,\dots,n$) . Then there exists a unique $v \in [P_1(K)]^n$ such that

(2.1) $$T_i v(a_{i_j}) = \gamma_j^{(i)} , \quad a_{i_j} \in \Sigma^i .$$

Proof. Let $a_i \in \Sigma$ be a vertex of the simplex K , and $S_{i_1}, S_{i_2},\dots,S_{i_n}$ the $(n-1)$-dimensional sides of K , containing a_i . We choose the equations from (2.1), concerning the vertex a_i only:

(2.2) $$v(a_i) \cdot n^{(i_j)} = \gamma_i^{(i_j)}$$

(suppose that a_i represents the i-th element in every Σ^{i_j}). As the vectors $n^{(i_j)}$, $j = 1,\dots,n$ are linearly independent, there exists a unique solution $v(a_i) = (v_1(a_i),\dots$ $\dots, v_n(a_i))$. From the $P_1(K)$-unisolvability of Σ it follows the existence of unique polynomials $v_j \in P_1(K)$, corresponding to the values $\{v_j(a_i)\}_{i=1}^{n+1}$, $j = 1,\dots,n$.

Let S_i be a $(n-1)$-dimensional side of K . We say that $\lambda_1^{(i)},\dots, \lambda_n^{(i)}$ are the basic functions of the side S_i, if

1° $\lambda_j^{(i)} \in P_1(S_i)$, $j = 1,\dots,n$,

2° $\lambda_j^{(i)}(a_{i_k}) = \delta_{jk}$, $a_{i_k} \in \Sigma^i$.

From the $P_1(S_i)$-unisolvability of every set Σ^i and from the definition of $\lambda_j^{(i)}$ it follows:

$$\sum_{j=1}^{n} \lambda_j^{(i)} = 1 \quad \text{on} \quad S_i \;. \tag{2.3}$$

An easy calculation leads to the relation

$$\int_{S_i} \lambda_j^{(i)} dS = \int_{S_i} \lambda_k^{(i)} dS \qquad \forall\, j,k = 1,\dots,n \;.$$

Consequently, using also (2.3) we derive that

$$\text{mes}_{n-1}(S_i) = \int_{S_i} \Big(\sum_{j=1}^{n} \lambda_j^{(i)} \Big) dS = n \,.\, \int_{S_i} \lambda_1^{(i)} \; dS \;,$$

$$\int_{S_i} \lambda_j^{(i)} \; dS = \frac{1}{n} \,\text{mes}_{n-1}(S_i) \;, \quad j = 1,\dots,n \;. \tag{2.4}$$

Henceforth we denote

$$(f,g) = \int_{S_i} f \, g \, dS$$

for any $f,\, g \in L_2(S_i)$.

<u>Theorem 2.1</u> Let $v \in [W^{1,2}(K)]^n$. Then the equations

$$\sum_{j=1}^{n} \alpha_j^{(i)} (\lambda_j^{(i)} ,\; \lambda_k^{(i)}) = (T_i v,\; \lambda_k^{(i)} \;) \;, \tag{$*$}$$

$$\Pi \; v(a_{i_k}) \,.\, n^{(i)} = \alpha_k^{(i)} \tag{$**$}$$

for $i = 1,\dots, n+1$; $k = 1,\dots,n$, $a_{i_k} \in \Sigma^i$

define a mapping $\Pi \in \mathcal{L}\,([W^{1,2}(K)]^n ,\; [P_1(K)]^n) \cap$

$\cap\, \mathcal{L}\,([C(K)]^n \,, [P_1(K)]^n)$.

<u>Proof</u>. The numbers $\alpha_k^{(i)}$ are uniquely determined by

$(*)$, because the matrix (Gramm's) A_i with entries

$$(A_i)_{st} = (\lambda_s^{(i)}, \lambda_t^{(i)}), \quad s, t = 1,2,\dots,n$$

is regular. Solving the system $(*)$ we obtain altogether $(n+1) \cdot n$ parameters $\alpha_k^{(i)}$. Lemma 2.1 yields the existence and uniqueness of a vector $\varphi \in [P_1(K)]^m$, for which

$$T_i \varphi (a_{i_j}) = \alpha_j^{(i)} .$$

We set $\Pi v = \varphi$. Obviously, the mapping Π is linear. Let $\alpha_{j,n}^{i}$ and $\alpha_j^{(i)}$ be the solution of $(*)$ with the right hand sides $(T_i v_n, \lambda_k^{(i)})$ and $(T_i v, \lambda_k^{(i)})$, respectively, $(k = 1,\dots,n)$ and let $v_n \to v$ in $[W^{1,2}(K)]^m$. From the theorem of traces (see [2]) and the Crammer's rule

$$\lim_{n \to \infty} \alpha_{j,n}^{(i)} = \alpha_j^{(i)}$$

follows. The rest of the proof is obvious.

Consequently, there exists a constant $c > 0$ such that

$$\| \Pi v \|_{C(K)} \leq c \, \| v \|_{C(K)} .$$

The magnitude of c will be estimated in the following

<u>Theorem 2.2</u> Let us define Π by $(*)$ and $(**)$. Then

$$\| \Pi v \|_{C(K)} \leq \frac{c_0}{\min \{ \det (m^{(i_1)}, \dots, m^{(i_n)}) \}} \| v \|_{C(K)}$$

where c_0 is an absolute constant and the minimum is taken over the set of all $n+1$ n-tuples of numbers $(i_1,\dots,i_n)$, chosen from the set $\{1,\dots,n+1\}$.

<u>Proof.</u> Let $\hat{K}$ represent the reference n-simplex in R_n, with the vertices $(0,\dots,0)$, $(1,0,\dots,0)$, ..., $(0,\dots,1)$.

Let $F(\hat{x}) = B\hat{x} + b$ be a regular affine mapping, of $\hat{K}$ onto K (see [3]). Then $S_i = F(\hat{S})$, where $\hat{S}$ is a $(n-1)$-dimensional side of $\hat{K}$. Using the integral mean value theorem we obtain

$$\int_{S_i} \lambda_j^{(i)} \lambda_k^{(i)} dS = J \cdot \int_{\hat{S}} \hat{\lambda}_j^{(i)}(\hat{x}) \lambda_k^{(i)}(\hat{x}) d\hat{S} =$$
$$= J \cdot \hat{\lambda}_j^{(i)}(\hat{\xi}_j^i) \cdot \hat{\lambda}_k^{(i)}(\hat{\xi}_k^i) \cdot mes(\hat{S}) =$$
$$= \hat{\lambda}_j^{(i)}(\hat{\xi}_j^i)\, \hat{\lambda}_k^{(i)}(\hat{\xi}_k^i)\, mes(S_i)$$

where $\lambda_j^{(i)}(\hat{x}) = \lambda_j^{(i)}(F\ \hat{x})$, $\hat{x} \in \hat{S}$, $\hat{\xi}_j^i \in \hat{S}$ and $J = \mathrm{const} > 0$ is the Jacobian of the transformation $\hat{S} \leftrightarrow S_i$.

Hence

$$(\lambda_j^{(i)}, \lambda_k^{(i)}) = c_{jk}^{(i)}\, mes(S_i),$$

with a constant $c_{jk}^{(i)}$ independent of S_i. Consequently,

$$\det A_i = \bar{c}_i (mes(S_i))^n, \quad \bar{c}_i \neq 0,$$

where $\bar{c}_i$ is a linear combination of product of $c_{jk}^{(i)}$.

Using the Crammer's rule and a similar estimate of the determinant in the numerator, we are led to the estimate

$$(2.5) \qquad |\alpha_j^{(i)}| \leqq c_i \|v\|_{C(K)},$$

where c_i is an absolute constant. Solving the system (**) we get $\Pi\, v(a_i) = (\varphi_1(a_i), \ldots, \varphi_n(a_i))$. Using again the Crammer's rule and (2.5), we obtain

$$|\varphi_j(a_i)| = \frac{c_0}{\min\{|\det(n^{(i_1)}, \ldots, n^{(i_n)})|\}} \|v\|_{C(K)},$$

where c_0 is an absolute constant and the denominator is defined in the Theorem. As $\Pi\, v \equiv \varphi \in [P_1(K)]^n$, the assertion follows immediately from the last inequality.

Let us denote

$$\mathcal{M}(K) = \{ v = (v_1,\dots,v_n) \in [P_1(K)]^n , \ \operatorname{div} v = 0 \} .$$

It is readily seen that $\dim \mathcal{M}(K) = (n+1) \cdot n - 1$.

<u>Lemma 2.2</u> $\quad v \in \mathcal{M}(K) \Longleftrightarrow v \in [P_1(K)]^n \ \&$

$$\& \int_{\partial K} v \cdot n \, dS = 0 .$$

<u>Proof</u>. Let $v \in [P_1(K)]^n$. Then $\operatorname{div} v \in P_0(K)$ and

$$\operatorname{div} v = 0 \Longleftrightarrow \int_K \operatorname{div} v \, dx = 0 .$$

Hence the Green's theorem

$$\int_K \operatorname{div} v \, dx = \int_{\partial K} v \cdot n \, dS$$

yields the assertion.

<u>Lemma 2.3</u> $\quad v \in \mathcal{M}(K) \Longleftrightarrow v \in [P_1(K)]^n \ \&$

$$\& \sum_{i=1}^{n+1} \sum_{k=1}^{n} \alpha_k^{(i)} \operatorname{mes}(S_i) = 0 ,$$

where $\alpha_k^{(i)} = T_i \, v(a_{i_k})$, $a_{i_k} \in \Sigma^i$.

<u>Proof</u>. We have

$$\int_{\partial K} v \cdot n \, dS = \sum_{i=1}^{n+1} \int_{S_i} T_i \, v \, dS , \quad T_i v = \sum_{k=1}^{n} \alpha_k^{(i)} \lambda_k^{(i)} .$$

Using also (2.4), we may write

$$\int_{\partial K} v \cdot n \, dS = \frac{1}{n} \sum_{i=1}^{n+1} \sum_{k=1}^{n} \alpha_k^{(i)} \operatorname{mes}(S_i) ,$$

and the assertion follows from Lemma 2.2.

Let us define

$$V(K) = \{ v \in [W^{1,2}(K)]^n , \ \operatorname{div} v = 0 \} .$$

Theorem 2.3 Let the mapping Π be defined by $(*)$ and $(**)$. Then

$$\Pi \in \mathcal{L}(V(K),\ \mathcal{M}(K)) ,$$

$$\Pi v = v \qquad \forall\, v \in [P_1(K)]^n . \tag{2.6}$$

Proof. Adding the equations $(*)$ for $k = 1,\dots,n$ we get

$$(T_i v, \sum_{k=1}^{n} \lambda_k^{(i)}) = \sum_{j=1}^{n} \alpha_j^{(i)} (\lambda_j^{(i)}, \sum_{k=1}^{n} \lambda_k^{(i)}) .$$

From (2.3) and (2.4) it follows

$$\int_{S_i} T_i v \, dS = \sum_{j=1}^{n} \alpha_j^{(i)} (\lambda_j^{(i)}, 1) = \frac{1}{n} \sum_{j=1}^{n} \alpha_j^{(i)} \operatorname{mes}(S_i) .$$

If $v \in V(K)$, then we have

$$0 = \int_{\partial K} v . n \, dS = \sum_{i=1}^{n+1} \int_{S_i} T_i v \, dS = \frac{1}{n} \sum_{i=1}^{n+1} \sum_{j=1}^{n} \alpha_j^{(i)} \operatorname{mes}(S_i) .$$

As $\alpha_j^{(i)} = T_i \Pi v(a_{i_j})$, $a_{i_j} \in \Sigma^i$, Lemma 2.3 yields $\Pi v \in \mathcal{M}(K)$. The assertion (2.6) is an immediate consequence of Lemma 2.1, because for $v \in [P_1(K)]^n$

$$\alpha_k^{(i)} = T_i\, v(a_{i_k}) \equiv v(a_{i_k}) . n^{(i)} .$$

Theorem 2.4 Let $v \in [C^2(K)]^n$ and $h = \operatorname{diam} K$. Then

$$\| v - \Pi v \|_{C(K)} \leqq \frac{c}{\min\{ |\det(n^{(i_1)},\dots,n^{(i_n)})| \}} h^2 \| v \|_{C^2(K)} \tag{2.7}$$

where c is an absolute constant and the denominator was defined in Theorem 2.2.

Proof. Let $x_0 \in K$ be an arbitrary fixed point. Using the Taylor's theorem we obtain

(2.8) $v(x) = v(x_0) + Dv(x_0)(x - x_0) + D^2v(\theta)(x - x_0)^2$,

with $\theta \in \overline{x_0 x}$. Applying Π to (2.8), using its linearity and (2.6), we derive

$$\Pi\, v(x) = v(x_0) + Dv(x_0)(x - x_0) + \Pi\ \ D^2v(\theta)(x - x_0)^2) \ .$$

Hence we may write

$$\| v - \Pi\, v \|_{C(K)} \leqq \| D^2v(\theta)(x - x_0)^2 \|_{C(K)} + $$
$$+ \ \ \| \Pi\ D^2v(\theta)(x - x_0)^2 \|_{C(K)} \ .$$

The estimate (2.7) follows from Theorem 2.2 and $\| x - x_0 \| \leqq$ $\leqq h$.

<u>Remark 2.1</u> For $n = 2$ we have

$$|\det(n^{(i_1)}, n^{(i_2)})| = |n^{(i_1)} \times n^{(i_2)}| = \sin \alpha_{12} \ ,$$

where α_{12} is the angle between vectors $n^{(i_1)}$, $n^{(i_2)}$. Hence

$$\| v - \Pi v \|_{C(K)} \leqq \frac{c}{\sin \alpha} h^2 \| v \|_{C^2(K)} \ ,$$

where α is the minimal angle of the triangle K .

For $n = 3$ there holds

$$|\det(n^{(i_1)}, n^{(i_2)}, n^{(i_3)})| = |n^{(i_1)} \cdot (n^{(i_2)} \times n^{(i_3)})| \ ,$$

which equals the volume of the parallelepiped, being determined by the three (unit) vectors $n^{(i_1)}$, $n^{(i_2)}$ and $n^{(i_3)}$.

Let us consider a bounded polyhedral domain $\Omega \subset R_n$. Let h be a parameter, $h \in (0.1 \rangle$, $\mathcal{T}_h$ a finite division of $\overline{\Omega}$, satisfying the usual conditions concerning the mutual position of any couple of n-simplexes $K, K' \in \mathcal{T}_h$, $h =$

$= \max_{K \in \mathcal{T}_h} (\operatorname{diam} K)$.

Let K and K' have a common $(n-1)$-dimensional side S_i , $v \in [W^{1,2}(\Omega)]^n$. Denote

$$T_{i,K}v = v|_{S_i} \cdot n_K^{(i)} , \quad T_{i,K'}v = v|_{S_i} \cdot n_{K'}^{(i)} ,$$

where $n_K^{(i)}$ and $n_{K'}^{(i)}$, is the unit outward normal with respect to K and K' , respectively, at a point $x \in S_i$.

We say that the condition (R) is satisfied on S_i , if

$$(2.9) \qquad T_{i,K}v + T_{i,K'}v = 0 \quad \text{on } S_i .$$

Denote

$$V(\Omega) = \{v \in [W^{1,2}(\Omega)]^n , \ \operatorname{div} v = 0\} ,$$

$$\mathcal{N}_h(\Omega) = \{v, v|_K \in \mathcal{M}(K), \ \forall K \in \mathcal{T}_h , \text{ and (R) is satisfied on every common side in } \mathcal{T}_h\} .$$

Let us define a mapping r_h of $[W^{1,2}(\Omega)]^n$ as follows:

$$(2.10) \qquad r_h v|_K = \Pi_K v \qquad \forall K \in \mathcal{T}_h$$

where Π_K has been defined in Theorem 2.1 through (*) and (**) on every $K \in \mathcal{T}_h$.

We say that a family of division $\{\mathcal{T}_h\}$, $h \in (0,1\rangle$ is regular, if a constant $\alpha_0 > 0$ exists such that for every $K \in \mathcal{T}_h$ and any $h \in (0,1\rangle$

$$(2.11) \qquad \min\{|\det (n_K^{(i_1)}, \dots, n_K^{(i_n)}|\} \geq \alpha_0 ,$$

the minimum being defined in Theorem 2.2.

The condition (2.11) implies for $n = 2$ and $n = 3$ that the corresponding angles and volumes, respectively, cannot converge to zero with $h \to 0$ (see Remark 2.1).

Theorem 2.5 Let $\{\mathcal{T}_h\}$, $h \in (0,1\rangle$ be a regular family of division. Define the mapping r_h as in (2.10). Then

$$r_h \in \mathcal{L}(V(\Omega), \mathcal{N}_h(\Omega)) , \tag{2.12}$$

$$\| v - r_h v \|_{0,\Omega} \leq ch^2 \| v \|_{C^2(\bar{\Omega})} . \tag{2.13}$$

Proof. From Theorem 2.3 it follows

$$r_h v|_K \in \mathcal{M}(K) \quad \forall v \in V(\Omega), \quad \forall K \in \mathcal{T}_h .$$

Hence it suffices to verify the condition (R) on every common $(n-1)$-dimensional side S_i in $\mathcal{T}_h$. As $T_{i,K}(\Pi_K v)$ and $T_{i,K'}(\Pi_{K'} v)$ belong to $P_1(S_i)$, it suffices to verify that

$$T_{i,K}(\Pi_K v)(a_{i_j}) + T_{i,K'}(\Pi_{K'} v)(a_{i_j}) = 0 \quad \forall a_{i_j} \in \Sigma^i \subset S_i .$$

This follows from (*), (**), because $n_K^{(i)} = - n_{K'}^{(i)}$, and therefore

$$T_{i,K}(\Pi_K v)(a_{i_j}) = \alpha_j^{(i)}$$

$$T_{i,K'}(\Pi_{K'} v)(a_{i_j}) = - \alpha_j^{(i)} .$$

The estimate (2.13) can be obtained in a usual way:

$$\| v - r_h v \|^2_{0,\Omega} = \sum_{K \in \mathcal{T}_h} \| v - \Pi_K v \|^2_{0,K} \leq ch^4 \| v \|^2_{C^2(\bar{\Omega})} ,$$

where (2.7) and (2.11) have been employed.

Remark 2.2 Any $v \in \mathcal{N}_h(\Omega)$ satisfies the equation $\operatorname{div} v = 0$ in the sense of distributions.

In fact, let $\varphi \in \mathcal{D}(\Omega)$ (i.e., an infinitely differentiable function with compact support). Then

$$\langle \operatorname{div} v, \varphi \rangle = - \langle v, \operatorname{grad} \varphi \rangle = - \sum_{K \in \mathcal{T}_h} \int_K v \cdot \operatorname{grad} \varphi \, dx =$$

$$= \sum_{K \in \mathcal{T}_h} \int_K \varphi \cdot \operatorname{div} v \, dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} T_K v \cdot \varphi \, dS = 0 ,$$

because $\operatorname{div} v = 0$ in every $K \in \mathcal{T}_h$ and the condition (R) is satisfied on every interelement boundary $\partial K \cap \partial K' = S_i$. We say that $\mathcal{T}_h$ is consistent with Γ_g, if the following condition holds: if a part of Γ_g belongs to a side S_i of a $K \in \mathcal{T}_h$, then Γ_g covers the whole side S_i.

It is easy to verify by a similar way that

$$V_h = \mathcal{N}_h(\Omega) \cap \Lambda_{0,0} = \{ \lambda \mid \lambda \in \mathcal{N}_h(\Omega), \lambda \cdot n = 0 \text{ on } \Gamma_g \}$$

defines the subspace V_h of H_2. Then Theorems 1.2 and (2.13) lead to the following

Theorem 2.7 Let the solution χ^0 of (1.4) belong to $C^2(\bar{\Omega})$ and $\{\mathcal{T}_h\}$, $h \in (0,1\rangle$ be a regular family of divisions consistent with Γ_g. Then

$$\| \chi^0 - \chi^0_h \|_{0,\Omega} = O(h^2) , \quad h \to 0$$

where χ^0_h is the solution of (1.5).

Remark 2.3 The principle of complementary energy (Theorem 1.1) can be extended to the mixed boundary-valued problem including the Newton's condition (see [1] - Appendix). The same subspaces $\mathcal{N}_h(\Omega)$ are applicable and an analogue of Theorem 2.7 holds.

Remark 2.4 As V_h belong to $\Lambda_{0,0} = H_2$, the dual method described above can be used to get (i) a posteriori error estimates and (ii) the solutions by the method of hypercircle (cf. [1]).

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