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FREE ALGEBRAS, INPUT PROCESSES AND FREE MONADS

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Abstract: Given a functor $F: \mathcal{X} \longrightarrow \mathcal{X}$, a category of F-algebras is defined and the existence of free F-algebras is discussed. This yields, under general conditions a characterization of input processes in the sense of Arbib, Manes or of free monads in the sense of Barr. The characterization is very simple: if F preserves monics then it is an input process iff for every object A there exists an object B with $B = A \vee FB$.

Key words: Free algebra, functor algebra, input process, free monad.

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A characterization of input processes or free monads has been a problem for a considerable time. For the category of sets a more general problem was solved by [Kůrková-Pohlová, Koubek] ; the present proof (of a special case) is simpler than theirs. The condition $B = A \vee FB$ originates from [Manes] .

The paper also answers positively the question, whether, whenever free algebras exist, they can be obtained by a standard construction (see [Adámek$_1$], where the construction for sets from [Pohlová] ,[Pohlová, Adámek, Koubek] is generalized).

I. Free F-algebras and free monads

I,1 Given a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$, define a category $\mathcal{K}(F)$ of F-algebras, i.e. pairs (X,d) , where X is a $\mathcal{K}$-object and $d: FX \longrightarrow X$ is a $\mathcal{K}$-morphism) the morphisms from the F-algebra (X,d) to the F-algebra (X',d') are those $\mathcal{K}$-morphisms $p: X \longrightarrow X'$, for which the diagram

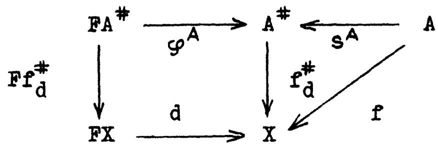
$$\begin{array}{ccc} FX & \xrightarrow{\ d\ } & X \\ {\scriptstyle Fp}\downarrow & & \downarrow{\scriptstyle p} \\ FX' & \xrightarrow{\ d'\ } & X' \end{array}$$

commutes.

There is a natural forgetful functor $U = U_F: \mathcal{K}(F) \longrightarrow \mathcal{K}$, assigning to each F-algebra (X,d) the object X .

No current notation of these categories has been adopted: [Barr] denotes them $(F:\mathcal{K})$; [MacLane] denotes them $(F \downarrow 0)$, where 0 is the initial object of $\mathcal{K}$; [Arbib, Manes] denote them Dyn (F) ; for $\mathcal{K}$ = Set , they are denoted by $A(F, 1)$ in [Adámek, Koubek, Pohlová] , [Kůrková-Pohlová, Koubek] , [Trnková, Goralčík] .

I,2 Let A be an object of $\mathcal{K}$. By a free F-algebra over A we mean an algebra $(A^{\#}, \varphi^{A})$ and a $\mathcal{K}$-morphism $s^{A}: A \longrightarrow A^{\#}$ such that for every F-algebra (X,d) and every $\mathcal{K}$-morphism $f: A \longrightarrow X$ there exists a unique morphism $f^{\#}_{d} : (A^{\#}, \varphi^{A}) \longrightarrow (X,d)$ with $f = f^{\#}_{d} \cdot s^{A}$:



In other words, a free algebra over A is a universal arrow from A to the forgetful functor U: $\mathcal{K}(F) \longrightarrow \mathcal{K}$.

Proposition: The following is equivalent.

(i) F admits free algebras, i.e. $(A^{\#}, \varphi^{A})$ exists for every A

(ii) U is a right adjoint

(iii) U is monadic.

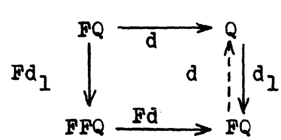
The proof of the above proposition is easy: (i) and (ii) are just re-formulation, (iii) $\longrightarrow$ (ii) follows easily from PTT (compare with the proof in [MacLane] for varieties of algebras).

Functors, which admit free algebras, are called input processes by [Arbib, Manes] . For such functors they build a model of automata theory in the category $\mathcal{K}$.

I,3 An F-algebra A is initial, if for every F-algebra A' there exists just one morphism from A to A' .

Lemma [Barr] : If (Q,d) is an initial F-algebra, d is an isomorphism.

Proof: There exists unique d_1: (Q,d) $\longrightarrow$ (FQ,Fd) ; then clearly $d \circ d_1$ is an endomorphism of (Q,d) , thus $d \circ d_1 = 1$



since $d_1 \circ d = F(d \circ d_1) = 1$, $d = (d_1)^{-1}$.

Assume that $\mathcal{K}$ has finite sums and, given A , denote by G the functor $GQ = FQ \vee A$.

<u>Lemma</u> [Manes] The F-algebra (A^+, φ) is free over A with $s: A \longrightarrow A^+$ iff the G-algebra (A^+, ψ) is initial, where $\psi : FA^+ \vee A \longrightarrow A^+$, $\psi = \varphi$ on FA^+ and $\psi = s$ on A .

<u>Proof</u> is easy.

<u>Corollary</u>. If $(A^{\#}, \varphi^A)$ is a free F-algebra over A , then $A^{\#} = A \vee FA^{\#}$.

This corollary was also proved by [Barr] .

I,4 By <u>a free monad</u>, generated by $F: \mathcal{K} \longrightarrow \mathcal{K}$ is meant a monad $\mathbb{T} = (T, \eta, \mu)$ over $\mathcal{K}$ for which there exists a transformation $\sigma : F \longrightarrow T$, universal in the following sense: for every monad $\mathbb{T}' = (T', \eta', \mu')$ over $\mathcal{K}$ and every transformation $\sigma' : F \longrightarrow T'$ there exists a unique monad morphism $\tau : \mathbb{T} \longrightarrow \mathbb{T}'$ with $\sigma' = \tau \circ \sigma$.

$\mathcal{K}$ is said to have <u>small</u> $\mathcal{M}$ <u>-factorizations</u> (where $\mathcal{M}$ is a class of monomorphism) if every morphism factorizes through some $\mathcal{M}$ -morphism and $\mathcal{K}$ is $\mathcal{M}$ -well-powered, i.e. every object has, up to isomorphism, only a set of $\mathcal{M}$ -subobjects.

<u>Theorem</u> [Barr] Let $\mathcal{K}$ be complete and cocomplete and have small $\mathcal{M}$ -factorizations. Then a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ generates a free monad iff it admits free algebras.

I,5 For a monad $\mathbb{T} = (T, \eta, \mu)$, $\mathcal{K}^{\mathbb{T}}$ denotes, as usual, the full subcategory of $\mathcal{K}(T)$ over monadic T-algebras which are those (Q,d) for which $d \circ \eta^Q = 1_Q$ and $d \circ \mu^Q = d \circ Td$.

<u>Definition</u>: The functor F is <u>a quasimonad</u> if there exists a monad $\mathbb{T}$ over $\mathcal{K}$ and an isomorphism $\psi : \mathcal{K}(F) \longrightarrow \mathcal{K}^{\mathbb{T}}$ which commutes with the forgetful functors:

$$\begin{array}{ccc} \mathcal{K}(F) & \xrightarrow{\ \psi\ } & \mathcal{K}^{\mathbb{T}} \\ & \searrow \quad \swarrow & \\ & \mathcal{K} & \end{array}$$

The following theorem is implicitly also stated in [Barr] . We present a proof, since it is easy and does not need any assumptions on $\mathcal{K}$.

<u>Theorem</u>. F is a quasimonad iff it admits free algebras.

<u>Proof</u>. Necessity. Define $\varphi^Q : FTQ \longrightarrow TQ$ by $\psi(TQ, \mu^Q) = (TQ, \varphi^Q)$. Then for every monadic T-algebra (Q,d) we have $\psi(Q,d) = (Q, d \circ \varphi^Q \circ F\eta^Q)$: indeed, $d : (TQ, \mu^Q) \longrightarrow (Q,d)$ is in $\mathcal{K}^{\mathbb{T}}$ and so $d : (TQ, \varphi^Q) \longrightarrow \psi(Q,d)$ is in $\mathcal{K}(F)$. Denote $\psi(Q,d) = (Q, \tilde{d})$. Thus, $\tilde{d} \circ Fd = d \circ \varphi^Q$.

$$\begin{array}{ccc} TTQ & \xrightarrow{\ \mu^Q\ } & TQ \\ {\scriptstyle Td}\downarrow & & \downarrow{\scriptstyle d} \\ TQ & \xrightarrow[\ d\ ]{} & Q \end{array} \qquad \begin{array}{ccc} FTQ & \xrightarrow{\ \varphi^Q\ } & TQ \\ {\scriptstyle F\eta^Q}\uparrow\downarrow{\scriptstyle Fd} & & \downarrow{\scriptstyle d} \\ FQ & \xrightarrow[\ \tilde{d}\ ]{} & Q \end{array}$$

Since $d \circ \eta^Q = 1_Q$, we have $\tilde{d} = d \circ \varphi^Q \circ F\eta^Q$. Now, (TA, φ^A) with $\eta^A : A \longrightarrow TA$ is a free F-algebra over

A. Indeed, every F-algebra $(Q,\tilde{d})$ has the form $\tilde{d} = d \circ \varphi^{Q} \circ F\eta^{Q}$ with (Q,d) in $\mathcal{K}^{\mathbb{T}}$; given $p: A \longrightarrow Q$, we have $p^{\#}_{\tilde{d}} = d \circ Tp$.

Sufficiency. Define the free-algebra monad $\mathbb{T}$: $TA = A^{\#}$, $\eta^{A} = s^{A}$, $\mu^{A} = (1_{A^{\#}})^{\#}_{\varphi^{A}} : A^{\#\#} \longrightarrow A^{\#}$. Define σ: $: F \longrightarrow T$, $\sigma^{A} = \varphi^{A} \circ Fs^{A}$. Then for every F-algebra (Q,d) we have $d^{+} = Q^{\#} \longrightarrow Q$, $d^{+} = (1_{Q})^{\#}_{d}$. It is easy to see that (Q,d^{+}) is in $\mathcal{K}^{\mathbb{T}}$ and that this defines a functor $\mathcal{K}(F) \longrightarrow \mathcal{K}^{\mathbb{T}}$. Its inverse functor is $(Q,e) \longrightarrow (Q, e \circ \varphi^{Q} \circ Fs^{Q})$.

II. Main theorem

II,1 Assume that $\mathcal{K}$ has finite sums and colimits of chains (i.e. of diagrams, the scheme of which is an ordinal). For an arbitrary object A define objects W_i and morphisms $s_{i,j}: W_j \longrightarrow W_i$ for arbitrary ordinals $j \leq i$ such that $s_{i,i} = 1$ and $s_{i,j} \circ s_{j,k} = s_{i,k}$ by <u>the free algebra construction</u>

<u>Free algebra construction</u> (for a given functor $F: \mathcal{K} \longrightarrow \mathcal{K}$):

$W_0 = A$

$W_{i+1} = A \vee FW_i$

a) i non-limit

$s_{1,0}: A \longrightarrow A \vee FA$ is canonical

$s_{i+1,i} = 1_A \vee Fs_{i,i-1}$

b) i limit

W_i and $s_{i,j}: W_j \longrightarrow W_i$ is the colimit of $(\{W_j\}_{j<i} ; \{s_{k,j}\}_{j \leq k < i})$

$s_{i+1,i}: W_i \longrightarrow W_{i+1}$ is the unique morphism with

$s_{i+1,i} \circ s_{i,0}: A \longrightarrow A \vee FW_i$ canonical, $s_{i+1,i} \circ s_{i,j+1} = 1_A \vee Fs_{i,j}$

The free algebra construction is said to stop (more in detail, to stop after α steps) if $s_{\alpha+1,\alpha}$ is an isomorphism. Then put $A^{\#} = W_{\alpha}$ and, for canonical $m: FW_{\alpha} \longrightarrow W_{\alpha+1}$, put $\varphi^A = (s_{\alpha+1,\alpha})^{-1} \circ m$. Then $(A^{\#}, \varphi^A)$ and $s^A = s_{\alpha,0}$ is a free algebra over A (see [Adámek$_1$]).

II,2 Definition. A functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ is non-excessive on an object A if there exists an object B with $B = A \vee FB$. A functor, non-excessive on all objects, is simply called non-excessive.

Examples. An endofunctor F of the category of sets is non-excessive iff for every cardinal α there exists a cardinal $\beta \geq \alpha$ with card $FX \leq$ card X for any set X with card $X = \beta$. (Originally this was the way how [Kůrková-Pohlová, Koubek] and [Pohlová] defined non-excessivity.) E.g., every small set-functor (i.e. a factorfunctor of a sum of a set of hom-functors) is non-excessive. The converse is not true: consider $FX = \{f: \alpha \longrightarrow X \mid \alpha$ is a singular cardinal, f is one-to-one$\} \cup \{0\}$, Fp, for $p: X \longrightarrow Y$ assigns to f either $p \circ f$ if $p \circ f \in FY$ or 0 if $p \circ f \notin FY$. Clearly, F is non-excessive but big [Koubek].

Also, for the category $\mathcal{K}$ of vector spaces over a field F is non-excessive iff for every cardinal α there exists a cardinal $\beta \geq \alpha$ with dim $FX \leq$ dim X whenever dim $X = \beta$ (or, equivalently, with card $FX \leq$ card X whenever card $X = \beta$).

II,3 From now on we shall assume that a category $\mathcal{K}$ with a class $\mathcal{M}$ of monics is given such that

A) all $\mathcal{K}$ -objects and all $\mathcal{M}$ -morphisms form a subcategory, say $\mathcal{L}$, of $\mathcal{K}$ which has finite sums and colimits of chains, both preserved by the embedding of $\mathcal{L}$ into $\mathcal{K}$.

B) $\mathcal{K}$ is $\mathcal{M}$ -well powered.

In fact, we shall not need the assumption that $\mathcal{M}$ is closed under composition so that the above conditions could be weakened by the corresponding way.

In current AB5 categories, the conditions are fulfilled for $\mathcal{M}$ = all monics. A lot of examples will be exhibited in the last section.

II,4 <u>Theorem</u>. Let $\mathcal{K}$, $\mathcal{M}$ satisfy A), B). If a functor $F: \mathcal{K} \longrightarrow \mathcal{K}$ preserves $\mathcal{M}$ (i.e. $F(\mathcal{M}) \subset \mathcal{M}$) then for each object A the following are equivalent.

(i) A free F-algebra over A exists,

(ii) the free algebra construction is defined and stops for A ,

(iii) F is non-excessive on A .

<u>Proof</u>: Since (ii) $\Longrightarrow$ (i) by [Adámek$_1$] and, by Corollary I,3, also (i) $\Longrightarrow$ (iii), it suffices to show that (iii) $\Longrightarrow$ (ii).

Denote by F' the restriction of F to $\mathcal{L}$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{F} & \mathcal{K} \\ \uparrow & & \uparrow \\ \mathcal{L} & \xrightarrow{F'} & \mathcal{L} \end{array} .$$

Now, let $(\{W_j\}_j, \{s_{i,j}\}_{j<i})$ be the chain from the free algebra construction over A with respect to $F' : \mathcal{L} \longrightarrow \mathcal{L}$. As $\mathcal{L} \hookrightarrow \mathcal{K}$ preserves finite sums and colimits of chains, we get easily from the commutativity of the above diagram that the chain coincides with that one over A with respect to F. Also, the non-excessivity of F' is equivalent to the non-excessivity of F. Thus, to prove the theorem, we have to show that the free algebra construction over A with respect to $F' : \mathcal{L} \longrightarrow \mathcal{L}$ stops provided that F' is non-excessive. Let B be given with $B = A \vee F'B$. We shall construct $\mathcal{M}$-morphisms $t_j : W_j \longrightarrow B$ by the transfinite induction. Let $t_0 : W_0 \longrightarrow B$ be the canonical embedding $A \longrightarrow$ $\longrightarrow A \vee F'B$. Let $t_j : W_j \longrightarrow B$ for $j < k$ be defined such that $t_i \circ s_{i,j} = t_j$ for every $j < i < k$. In case k is limit, W_k is a colimit of $(\{W_j\}_j, \{s_{i,j}\}_{j<i})$ and so we have a unique $\mathcal{M}$-morphism $t_k : W_k \longrightarrow B$ such that $t_k \circ s_{k,i} = t_i$ for $i < k$. If k is isolated, put $t_k = 1_A \vee F\, t_{k-1}$. It is to be easily checked that $t_k \circ s_{k,k-1} = t_{k-1}$ so that $t_k \circ s_{k,i} = t_i$ holds for every $i < k$. As $\mathcal{K}$ is $\mathcal{M}$-well powered, there are i, j, say $j < i$, such that $s_{i,j}$ is an isomorphism. Then $s_{j+1,j}$ is also an isomorphism, so the free algebra construction stops.

<u>Corollary</u>. If F preserves $\mathcal{M}$ then it is an input process (or a quasimonad) iff it is non-excessive.

<u>Corollary</u>. Let $\mathcal{K}$, in addition to A), B) be complete and cocomplete and have a small $\mathcal{M}$-factorization. Then a functor, preserving $\mathcal{M}$, generates a free monad iff it is non-excessive.

III. Examples

a) The category Vect of vector spaces (over a field) fulfils A), B) with $\mathcal{M}$ = all monics. Since monics in Vect split, each functor preserves them so that the main theorem characterizes input processes among all endofunctors:

Proposition. A functor F: Vect $\longrightarrow$ Vect admits free algebras iff it is non-excessive, i.e. iff for every space A there exists a space B with $\dim B \geq \max(\dim A, \dim FB)$.

b) The category Set of sets. Following the same reasoning as in the case of Vect , and endofunctor F' of the category Set$'$ of non-void sets admits free algebras iff it is non-excessive, i.e. for every set A there is a set B with card $B \geq \max$ (card A, card FB) .

Now, for F: Set $\longrightarrow$ Set either $FX = \emptyset$ for all X or there is a restriction F' : Set$'$ $\longrightarrow$ Set$'$ of F . Clearly, F is non-excessive iff F' is, and the free F-algebra over $A \neq \emptyset$ coincides with the free F'-algebra over A . Thus, a free F-algebra over $A \neq \emptyset$ exists iff F is non-excessive on A . Following [Adámek, Koubek, Pohlová] , the same is true for $A = \emptyset$ (the free F-algebra over $\emptyset$ is just the initial F-algebra, i.e. a cosingleton in the terminology of the paper referred to). Thus we get a characterization of all input processes in Set:

Proposition [Kůrková-Pohlová, Koubek] . A functor F: : Set $\longrightarrow$ Set admits free algebras iff it is non-excessive, i.e. iff for every set A there is a set B with card $B \geq$ $\geq \max$ (card A, card FB) .

c) Concrete categories with colimits preserving forgetful functor. If $\mathcal{X}$ is a concrete cocomplete category whose forgetful functor preserves monics and colimits then the assumptions of the main theorem are clearly fulfilled for $\mathcal{M}$ = all monics (in fact, we need only that $\mathcal{X}$ has finite sums and colimits of chains and that the forgetful functor preserves monics, finite sums and colimits of chains). Thus the main theorem characterizes input processes among all monics preserving endofunctors of the category of topological (uniform, proximity) spaces, of graphs, of partial finitary algebras (of a given type) and of varieties of unary algebras without definable constants.

d) Varieties of finitary algebras fulfil A), B) for $\mathcal{M}$ = all monics, varieties of unary algebras also for $\mathcal{M}$ = all summands (i.e. canonical maps $A \longrightarrow A \vee B$).

e) For the category of graphs and the categories of partial finitary algebras we get three independent corollaries of the main theorem by putting 1) $\mathcal{M}$ = all monics, 2) $\mathcal{M}$ = all extremal monics (= all equalizers in this case), 3) $\mathcal{M}$ = all summands.

f) The category of extremally disconnected compact Hausdorff spaces satisfies the assumptions of the main theorem for $\mathcal{M}$ = = all summands, i.e. homeomorphisms onto an open-and-closed subsets (but it does not do so for $\mathcal{M}$ = all monics = all extremal monics !). This is verified by the following lemma; note that a colimit of a chain of subspaces in this category is the Čech-Stone compactification of its union.

Lemma. Let A_i , $i < k$, be subspaces of an extremally disconnected compact Hausdorff space X and let A_j be an open-and-closed subspace of A_i for every $i \geq j$ and also of X . If $\bigcup_{i<k} A_i$ is dense in X then $X = \beta(\bigcup_{i<k} A_i)$.

Proof: Clearly, $\bigcup_{i<k} A_i$ is an open subspace of X . In an extremally disconnected compact Hausdorff space, each bounded real continuous function, defined on an open subset, can be continuously extended to its closure, see e.g. [Gillman, Jerison] .

Corollary. Let F be an endofunctor of the category of extremally disconnected compact Hausdorff spaces, let F preserve open-and-closed subspaces. Then, F has a free algebra over a space X iff F is non-excessive on X .

In particular, this is so if F preserves finite sums.

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