Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 16 (1975), No. 1, 147--149

Persistent URL: http://dml.cz/dmlcz/105612

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16,1 (1975)

MINIMAL CELL COVERINGS OF SPHERE BUNDLES OVER SPHERES

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Abstract: It is shown that every sphere bundle over a sphere admits a covering by three open (or closed) cells.

Key words: Fibre space, Ljusternik-Schnirelman category.

AMS: 55F05, 57A15 Ref. Ž.: 3.976.1, 3.985

Let M be the total space of a locally trivial fibre space $\pi : M \rightarrow S^p$ with base space S^p and fibre S^q, and let $n = p + q$. In [1], it was shown that M can be covered by three open n-cells, if the fibration admits a global cross-section. By exploiting the topological symmetry of M, we now find the cross-section hypothesis superfluous. For completeness, we commence with the following lemma which is well-known among students of geometric topology:

Lemma. *Let $\{D_1, \ldots, D_k\}$ be a finite collection of mutually disjoint sets, each of which is cellularly embedded in the interior of a topological manifold M of dimension n. Then there is a closed n-cell F in the interior* *of* M *which contains* $D_1 \cup \dots \cup D_k$.

Proof. (Suggested by Prof. J.G. Hocking.) It suffices to prove the statement for $k = 2$. Let $\eta : M \longrightarrow M/\{D_1, D_2\}$ be the projection map onto the quotient space. A tame arc can be passed through $\eta(D_1)$ and $\eta(D_2)$ in this space, and this arc possesses an n-cell neighborhood N . The sought-for cell F may be taken to be $\eta^{-1}(N)$.

Theorem. *Let* M *be the total space of a locally trivial fibre space* $\pi : M \longrightarrow S^p$ *with base space* S^p *and fibre* S^q , *and let* $n = p + q$. *Then* M *can be covered by three open (or closed)* n-*cells*.

Proof. We regard the base space S^p as the union of two closed hemispheres S_+ and S_- with common boundary S^{p-1} . Now $M_+ = \pi^{-1}(S_+)$ is homeomorphic with $I^p \times S^q$ (it is the total space of a fibration with contractible base space), and hence there is a local cross section $\sigma_+ : S_+ \longrightarrow M_+$. The removal from M_+ of a small open product neighborhood N_+ of the image of σ_+ yields a closed n-cell F_+ .

Turning our attention to M_- , we define a local cross-section $\sigma_- : S_- \longrightarrow M_-$ by requiring that

$$\sigma_-(x) = \alpha_x \sigma_+(x) \quad \text{for} \quad x \in S^{p-1}$$

and extending this map over S_- , using the linear structure of I^p and the product structure of M_- . (Here α_x denotes the antipodal map in the q-sphere fibre $\pi^{-1}(x)$.) The closed n-cell F_- is obtained precisely as was F_+ , and if the product neighborhoods N_+ and N_- of the images of σ_+ and σ_- have been chosen sufficiently small, they are necessarily separated by a positive distance in M.

Since the complement of $F_+ \cup F_-$ in M is merely $N_+ \cup N_-$, we need only find a closed n-cell which contains the latter set. The mode of definition guarantees that $\overline{N_+} \cap \overline{N_-} = \emptyset$, and that each of the sets $\overline{N_+}$, $\overline{N_-}$ is cellularly embedded in M. (This last assertion follows from the fact that the product structures on M_+ and M_- can be extended to neighborhoods of these sets in M.) Therefore, the conditions of the lemma are satisfied and there is a closed n-cell F in M which contains $N_+ \cup N_-$.

Then $M = F_- \cup F \cup F_+$.

All three of the closed cells F_-, F, F_+ being themselves cellularly embedded in M, no difficulty arises in enclosing them in open n-cell neighborhoods, if a covering of M by open cells is wanted.

Reference

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(Oblatum 4.11.1974)