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CONTINUITY OF NĚMYCKIJ'S OPERATOR IN HÖLDER SPACES

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Abstract: This paper deals with the investigation of the mapping $u(x) \longmapsto f(u(x))$, where f is a given real valued function. There are proved the necessary and sufficient conditions upon f to be $f(u(x))$ Hölder-continuous function for an arbitrary Hölder-continuous function u and, moreover, the necessary and sufficient conditions for the mapping considered to be continuous between the spaces of Hölder-continuous functions.

Key-words: Spaces of Hölder-continuous functions, Němyckij's operator.

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1. Introduction. Let M, N be positive integers. Denote by $\mathbb{R}^M$ and $\mathbb{R}^N$, respectively, the M-dimensional and N-dimensional, respectively, Euclidean spaces with the norms $\|\cdot\|_M$ and $\|\cdot\|_N$, respectively. Let Ω be an open bounded non-empty subset of $\mathbb{R}^N$. For $\alpha \in (0,1\rangle$ define $H_\alpha^M(\Omega)$ the (so-called Hölder) space of all mappings $u: \Omega \longmapsto \mathbb{R}^M$ defined on Ω and with values in $\mathbb{R}^M$ such that

$$\|u\|_{H_\alpha^M} = \sup_{x \in \Omega} \|u(x)\|_M + \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{\|u(x) - u(y)\|_M}{\|x - y\|_N^\alpha} < +\infty$$

It is easy to see that $H_\alpha^M(\Omega)$ is a Banach space with the norm $\|\cdot\|_{H^M_\alpha}$. Instead of $H^1_\alpha(\Omega)$ we shall write $H_\alpha(\Omega)$ only.

Let $f: \mathbb{R}^M \longrightarrow \mathbb{R}^1$ be a real valued function. For $u \in H^M_\alpha(\Omega)$ denote by $\mathcal{N}(u)$ the function defined on Ω by the relation

$$\mathcal{N}(u)(x) = f(u(x)) , \quad x \in \Omega .$$

The mapping $\mathcal{N}$ is usually called the Němyckij's operator. In this paper, we give the necessary and sufficient conditions upon f to be $\mathcal{N}(u) \in H_\alpha(\Omega)$ for any $u \in H^M_\alpha(\Omega)$ (see Theorem 1) and also the necessary and sufficient conditions upon f to be the mapping $\mathcal{N}$ continuous from the space $H^M_\alpha(\Omega)$ into $H_\alpha(\Omega)$ (see Theorem 2). It is interesting that if $\mathcal{N}$ works from $H^M_\alpha(\Omega)$ into $H_\alpha(\Omega)$, then it is not generally continuous. This is a quite different result than for the space $C(\overline{\Omega})$ of continuous functions or for the space $L_p(\Omega)$ of p-integrable measurable functions (see e.g. [1],[2]).

2. Necessary and sufficient conditions for $\mathcal{N}(H^M_\alpha(\Omega)) \subset H_\alpha(\Omega)$.

Let $\mathbb{N}$ denote the set of all positive integers.

Lemma 1. **Let** $\alpha \in (0,1\rangle$ and let $\{a^i_n\}$, $i = 1,2,\dots, M$, **be** the real convergent sequences. Then for each $\varepsilon \in (0,1)$ there exists an increasing function k defined on $\mathbb{N}$ with values in $\mathbb{N}$ and an increasing sequence $\{t_n\} \in \langle 0, \varepsilon \rangle$ such that for each $m, n \in \mathbb{N}$ it is

$$\sum_{i=1}^{M} |a^i_{k(n)} - a^i_{k(m)}| \leq |t_n - t_m|^{\alpha} .$$

Proof. Let $\varepsilon > 0$. If $n \in \mathbb{N}$ then there exists $k(n)$ such that for each $m > k(n)$ and $i = 1,2,\dots, M$ it is

$$\sum_{i=1}^{M} |a^i_{k(n)} - a^i_{k(m)}| < \frac{\varepsilon}{2^n} .$$

It is possible to construct k to be an increasing function. Put

$$t_n = \sum_{i=1}^{n-1} \frac{\varepsilon}{2^i} .$$

Now, one can immediately see that t_n is the wanted sequence.

Lemma 2. Let the operator $\mathcal{N}$ map $H^M_\alpha(\Omega)$ into $H_\alpha(\Omega)$. Then f is a continuous function on $\mathbb{R}^M$.

Proof. Let us suppose that f is not continuous at the point $a_o = [a^1_o, \dots, a^M_o] \in \mathbb{R}^M$.

Then there exists $\omega_0 > 0$ and a sequence $a_n = [a^1_n, \dots, a^M_n] \in \mathbb{R}^M$ $(n = 1,2,\dots)$ such that $\lim_{n\to\infty} \| a_n - a_o \|_M = 0$ and

(1) $$|f(a_n) - f(a_o)| \geq \omega_0$$

for all $n \in \mathbb{N}$.

Choose $x_0 = [x_0^1, \dots, x_0^N] \in \Omega$ arbitrary but fixed and let $K_{2\varepsilon}(x_0) \subset \Omega$ where $K_{2\varepsilon}(x_0)$ is a ball centered at x_0 and with the radius 2ε .

Let $\{t_n\}$ and k have the same meaning as in Lemma 1. Put

$$x_n = x_0 + [t_n, 0, \dots, 0] = [x_0^1 + t_n, x_0^2, \dots, x_0^N] .$$

Obviously $x_n \in K_\varepsilon(x_0), (n \in \mathbb{N})$ and $\{x_n\}$ converges to some $z \in K_{2\varepsilon}(x_0)$. For each $n, m \in \mathbb{N}$ we have

$$(2) \quad \| a_{k(n)} - a_{k(m)} \|_M = \sum_{i=1}^{M} | a^i_{k(n)} - a^i_{k(m)} | \le$$

$$\le |t_n - t_m|^\alpha = \| x_n - x_m \|_N^\alpha .$$

Let us define

$$u(x_n) = a_{k(n)} , \quad n = 1,2,\dots; \quad u(z) = a_0 ,$$

and denote $\mathfrak{M} = \{x_n\} \cup \{z\}$. According to (2) u is bounded on the closed set $\mathfrak{M}$ and satisfies the Hölder condition on $\mathfrak{M}$. Thus (see e.g. [3, Proposition 1]) there exists a function U defined on $\bar{\Omega}$ such that $U \in H^M_\alpha(\Omega)$, the restriction of U on $\mathfrak{M}$ is u ,

$$\sup_{x \in \Omega} \| U(x) \|_M = \sup_{x \in \mathfrak{M}} \| u(x) \|_M ,$$

and

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{\|\mathcal{U}(x)-\mathcal{U}(y)\|_M}{\|x-y\|_N^{\alpha}} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{\|u(x)-u(y)\|_M}{\|x-y\|_N^{\alpha}} .$$

So $g \in H_{\alpha}(\Omega)$, where

$$g\colon x \longmapsto f(U(x)),\ x \in \bar{\Omega} .$$

It means that g is a continuous function on Ω , especially,

$$f(a_0) = g(a_0) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(u(x_n)) = \lim_{n\to\infty} f(a_{k(n)}),$$

which is a contradiction with (1).

Theorem 1. The operator $\mathcal{N}$ maps $H_{\alpha}^{M}(\Omega)$ into $H_{\alpha}(\Omega)$ if and only if f is a locally lipschitzian function from $\mathbb{R}^M$ into $\mathbb{R}$.

Proof. If f is locally lipschitzian, then we immediately obtain $\mathcal{N}(H_{\alpha}^{M}(\Omega)) \subset H_{\alpha}(\Omega)$.

On the other hand, let us suppose that $\mathcal{N}$ maps $H_{\alpha}^{M}(\Omega)$ into $H_{\alpha}(\Omega)$ and that f is not a locally lipschitzian function. Then there exist bounded sequences $\{\xi_n\}$, $\{\eta_n\} \subset \mathbb{R}^M$ such that

$$|f(\xi_n) - f(\eta_n)| > n\,\|\xi_n - \eta_n\|_M$$

for $n \in \mathbb{N}$. We can suppose that $\lim_{n\to\infty} \xi_n = \xi_0$ and $\lim_{n\to\infty} \eta_n = \eta_0$. It is $\xi_0 = \eta_0$, for Lemma 2 implies

$$|f(\xi_n) - f(\eta_n)| \leq \text{constant}$$

and thus

$$\text{constant} > n \, \| \xi_m - \eta_m \|_M$$

for each $n \in \mathbb{N}$.

In order that the following construction be simpler, we can suppose (without loss of generality) that Ω is an open ball in $\mathbb{R}^N$ centered at the origin and the radius of which is $r > 0$.

Consider an open ball $K_k \subset \mathbb{R}^M$, $k \in \mathbb{N}$, $\frac{r}{2^{k+1}} < 1$ centered in ξ_0 and with radius $\frac{r}{2^{k+2}}$. There exists $n_1 \in \mathbb{N}$ so that ξ_{m_1} and η_{m_1} are situated in K_k (denote $\xi_{m_1} = \xi_1^1$ and $\eta_{m_1} = \eta_1^1$). Denote $A_1 = \{[t_1,0,\ldots,0] \mid t \in \mathbb{R}^1\}$ and $x_1 = [0,0,\ldots,0] \in \mathbb{R}^N$. Let $y_1 \in A_1$, and $y_1 = [t_1,0,\ldots,0]$, $t_1 > 0$ and, moreover,

$$\| \xi_1^1 - \eta_1^1 \|_M = \| x_1 - y_1 \|_N^\alpha \quad .$$

It is obvious that

$$\| x_1 - y_1 \|_N < \frac{r}{2^{k+1}} \quad .$$

Further put

$$K_{k+1} = \{ z \in \mathbb{R}^M \mid \| z - \xi_0 \|_M < \frac{1}{2^{k+3}} \} .$$

There exist $\xi_2^1 \in \{\xi_m\}$, $\eta_2^1 \in \{\eta_m\}$ such that ξ_2^1 ,

$\eta_2^1 \in K_{k+1}$. Let $x_2 \in A_1$, $y_2 \in A_1$ be such that $x_2 =$ $= [\tau_2, 0, \ldots, 0] \in \mathbb{R}^N$, $\tau_2 - t_1 = \frac{\varkappa}{2^{k+1}}$, $y_2 =$ $= [t_2, 0, \ldots, 0] \in \mathbb{R}^N$, $t_2 > \tau_2$ and $\| \xi_2^1 - \eta_2^1 \|_M =$ $= \| x_2 - y_2 \|_N^\infty$.

Obviously, $\| x_2 - y_2 \|_N < \frac{\varkappa}{2^{k+2}}$.

Consider an open ball $K_{k+m} = \{ z \in \mathbb{R}^M \mid \| z - \xi_0 \|_M < \frac{1}{2^{k+m+2}} \}$. There exist $\xi_m^1 \in \{\xi_m\}$, $\eta_m^1 \in \{\eta_m\}$ such that $\xi_m^1 \in K_{k+m}$, $\eta_m^1 \in K_{k+m}$. Let $x_m \in A_1$, $y_m \in A_1$, be such that $x_m = [\tau_m, 0, \ldots, 0] \in \mathbb{R}^N$, $\tau_m - t_{m-1} =$ $= \frac{\varkappa}{2^{k+m}}$, $y_m = [t_m, 0, \ldots, 0] \in \mathbb{R}^N$, $t_m > \tau_m$ and $\| \xi_m^1 - \eta_m^1 \|_M = \| x_m - y_m \|_N^\infty$.

Clearly $\| x_m - y_m \|_N < \frac{\varkappa}{2^{k+m+1}}$.

Let x_0 mean the limit of the sequence $\{z_n\}$, where $z_{2\ell} = x_\ell$, $z_{2\ell-1} = y_\ell$ ($\ell \in \mathbb{N}$). We have

$$\| x_0 \|_N < 2 \sum_{i=k}^{\infty} \frac{\varkappa}{2^{i+1}} \leq 2r \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} = r ,$$

which means that the whole sequence $\{z_n\}$ and also x_0 are situated in Ω. Let us define the function $u_1 = [u_1^1, u_1^2, \ldots, u_1^M]$ as follows:

$$u_1(x_m) = \xi_m^1 \quad , \quad m = 1,2,\dots ,$$

$$u_1(y_m) = \eta_m^1 \quad , \quad m = 1,2,\dots ,$$

$$u_1(x_o) = \xi_o \ .$$

Put $Z = \{z_n\}_{n=1}^{\infty} \cup \{x_o\}$. The set $Z \subset \mathbb{R}^N$ is closed and u_1 satisfies on Z the Hölder condition with the exponent α . Indeed, we can suppose (without loss of generality) that $m \leq n$ and thus:

a) $\| u_1(x_m) - u_1(x_n) \|_M = \| \xi_m^1 - \xi_n^1 \|_M < \frac{r}{2^{k+m}} <$

$< \| x_m - x_n \|_N$

and if $\| x_m - x_n \|_N \geq 1$ then $\frac{r}{2^{k+m}} \leq \| x_m - x_n \|_N^{\alpha}$,

and if $\| x_m - x_n \|_N < 1$ then $\| x_m - x_n \|_N <$

$< \| x_m - x_n \|_N^{\alpha}$;

b) $\| u_1(x_m) - u_1(y_n) \|_M = \| \xi_m^1 - \eta_m^1 \|_M < \frac{r}{2^{k+m}} <$

$< \| x_m - y_n \|_N$ for $m < n$;

for $m = n$ we have $\| \xi_m^1 - \eta_m^1 \|_M = \| x_m - y_m \|_N^{\alpha}$;

c) $\| u_1(x_m) - u_1(x_o) \|_M < \frac{r}{2^{k+m}} < \| x_m - x_o \|_N$

and further we use the same arguments as in a);

d) analogously as in c) we estimate

$$\| u_1(y_m) - u_1(x_0) \|_M < \frac{\varkappa}{2^{k+m}} .$$

By using [3, Proposition 1], there exist the extensions of u^i, $i = 1,\dots, M$, so that $u^i \in H_\alpha(\Omega)$ for $i = 1,\dots, M$. It means that $u = [u^1,\dots, u^M] \in H_\alpha^M(\Omega)$. But for each $K > 0$ there exists $m \in \mathbb{N}$ such that

$$|f(u(x_m)) - f(u(y_m))| = | f(\xi_m^1) - f(\eta_m^1)| \geq$$

$$\geq K \| \xi_m^1 - \eta_m^1 \|_M = K \| x_m - y_m \|_N^\alpha ,$$

and so $f \circ u \notin H_\alpha(\Omega)$. This contradiction completes the proof of Theorem 1.

3. Necessary and sufficient conditions for continuity of $\mathcal{N}$

Lemma 3. Let the partial derivatives of the first order of the function f be continuous on $\mathbb{R}^M$ and let σ be a bounded subset of $\mathbb{R}^M$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $a \in \sigma$ and $h \in \mathbb{R}^M$ with $0 < \| h \|_M < \delta$ it is

$$\left| \frac{f(a+h) - f(a)}{\| h \|_M} - \sum_{i=1}^{M} \frac{\partial f(a)}{\partial \xi_i} \cdot \frac{h_i}{\| h \|_M} \right| < \varepsilon .$$

Proof. The uniform continuity on σ of the partial derivatives implies that for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $a \in \mathcal{O}$, $h \in \mathbb{R}^M$ with $0 < \| h \|_M < \delta$, and $\theta_a \in (0,1)$ it is

$$\sum_{i=1}^{M} \frac{|h_i|}{\|h\|_M} \left| \frac{\partial f(a+\theta_a h)}{\partial \xi_i} - \frac{\partial f(a)}{\partial \xi_i} \right| < \varepsilon \quad ,$$

$$f(a+h) - f(a) = \sum_{i=1}^{M} \frac{\partial f(a+\theta_a h)}{\partial \xi_i} h_i \quad .$$

It means that

$$\varepsilon > \sum_{i=1}^{M} \frac{|h_i|}{\|h\|_M} \left| \frac{\partial f(a+\theta_a h)}{\partial \xi_i} - \frac{\partial f(a)}{\partial \xi_i} \right| \geq$$

$$\geq \left| \sum_{i=1}^{M} \frac{\partial f(a+\theta_a h)}{\partial \xi_i} \frac{h_i}{\|h\|_M} - \sum_{i=1}^{M} \frac{\partial f(a)}{\partial \xi_i} \frac{h_i}{\|h\|_M} \right| =$$

$$= \left| \frac{f(a+h) - f(a)}{\|h\|_M} - \sum_{i=1}^{M} \frac{\partial f(a)}{\partial \xi_i} \frac{h_i}{\|h\|_M} \right| \quad .$$

<u>Theorem 2.</u> The operator $\mathcal{N}$ is continuous from $H_\alpha^M(\Omega)$ into $H_\alpha(\Omega)$ if and only if the partial derivatives of the first order of the function f are continuous.

<u>Proof</u>. Let us suppose, at first, that the partial derivatives of the first order of the function f are continuous. This means that f is a locally lipschitzian func-

tion (if $\sigma \subset \mathbb{R}^M$ is bounded, let $K(\sigma)$ be such a positive number that

$$| f(\xi) - f(\eta) | \leq K(\sigma) \| \xi - \eta \|_M$$

for all $\xi, \eta \in \sigma$). According to Theorem 1 it is

$$\mathcal{N} (H_\alpha^M(\Omega)) \subset H_\alpha(\Omega) .$$

Now, let us prove the continuity of $\mathcal{N}$ in an arbitrary point $u_0 \in H_\alpha^M(\Omega)$. Let $\epsilon > 0$.
Denote

$$W(\Delta) = \{ u \in H_\alpha^M(\Omega) \mid \| u - u_0 \|_{H_\alpha^M} < \Delta \}$$

for $\Delta > 0$. Then we have

$$\frac{\|u_0(x)-u_0(y)\|_M}{\|x-y\|_N^\alpha} - \Delta < \frac{\|u(x)-u(y)\|_M}{\|x-y\|_N^\alpha} < \frac{\|u_0(x)-u_0(y)\|_M}{\|x-y\|_N^\alpha} + \Delta$$

for all $u \in W(\Delta)$, $x, y \in \Omega$, $x \neq y$, and thus there exists $K > 0$ such that

$$\| u(x) - u(y) \|_M \leq K \| x - y \|_N^\alpha$$

provided $u \in W(\Delta)$, $x, y \in \Omega$. This implies that there exists a bounded set $\sigma \in \mathbb{R}^M$ such that $u(x) \in \sigma$ for all $u \in W(\Delta)$ and $x \in \Omega$. Put

$$A(u) = \sup_{x \in \Omega} | f(u(x)) - f(u_0(x)) | .$$

Obviously

(3) $$A(u) \leq \frac{\varepsilon}{2}$$

if $u \in W(\Delta) \cap W\left(\frac{\varepsilon}{2K(\sigma)}\right)$.

We denote $\varepsilon_1 = \dfrac{\varepsilon}{8K(\sigma)}$,

$$B(u) = \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(u(x)) - f(u_0(x)) - (f(u(y)) - f(u_0(y)))|}{\|x - y\|_N^\alpha},$$

and

$$C(u,x,y) = \left| \frac{f(u(x)) - f(u(y))}{\|u(x) - u(y)\|_M} \cdot \frac{\|u(x) - u(y)\|_M}{\|x - y\|_N^\alpha} - \right.$$

$$\left. \frac{f(u_0(x)) - f(u_0(y))}{\|u_0(x) - u_0(y)\|_M} \cdot \frac{\|u_0(x) - u_0(y)\|_M}{\|x - y\|_N^\alpha} \right|$$

if $u(x) \neq u(y)$, $u_0(x) \neq u_0(y)$, $x \neq y$.

Thus

$$\| u(x) - u(y) - (u_0(x) - u_0(y))\|_M \leq \varepsilon_1 \; \| x - y \|_N^\alpha$$

for each $x, y \in \Omega$ and $u \in W(\Delta) \cap W\left(\frac{\varepsilon}{8K(\sigma)}\right)$. It is easy to see that if $u(x) = u(y)$ or $u_0(x) = u_0(y)$ then

$$\frac{|f(u(x)) - f(u_0(x)) - (f(u(y)) - f(u_0(y)))|}{\|x - y\|_N^\alpha} \leq \frac{\varepsilon}{8} .$$

Let $x, y \in \Omega$ and let

$h'(x,y) = u(x) - u(y) \neq 0$, $h(x,y) = u_0(x) - u_0(y) \neq 0$.

Sc we obtain

$$C(x,y,u) = \left| \frac{f(u(y) + h'(x,y)) - f(u(y))}{\| h'(x,y) \|_M} \cdot \frac{\| h'(x,y) \|_M}{\| x - y \|_N^\alpha} - \right.$$

$$\left. - \frac{f(u_0(y) + h(x,y)) - f(u_0(y))}{\| h(x,y) \|_M} \quad \frac{\| h(x,y) \|_M}{\| x - y \|_N^\alpha} \right| .$$

If $\varepsilon_2 = \dfrac{\varepsilon}{16K}$, then with respect to the assertion of Lemma 3 there exists $\delta_2 > 0$ such that for each $h', h \in \mathbb{R}^M$, $0 < \| h' \|_M < \delta_2$, $0 < \| h \|_M < \delta_2$ we have

$$\left| \frac{f(u(y) + h') - f(u(y))}{\| h' \|_M} - \right.$$

$$\left. - \sum_{i=1}^{M} \cdot \frac{\partial f(u(y))}{\partial \xi_i} \cdot \frac{h'_i}{\| h' \|_M} \right| < \varepsilon_2 \quad ,$$

(4)

$$\left| \frac{f(u_0(y) + h) - f(u_0(y))}{\| h \|_M} - \right.$$

$$\left. - \sum_{i=1}^{M} \cdot \frac{\partial f(u_0(y))}{\partial \xi_i} \cdot \frac{h_i}{\| h \|_M} \right| < \varepsilon_2 \quad .$$

If $\| x - y \|_N^\alpha < \dfrac{\delta_2}{K}$ then $\| h'(x,y) \|_M < \delta_2$, $\| h(x,y) \|_M < \delta_2$.

The uniform continuity of partial derivatives of the function f on $\mathcal{O}$ implies the existence of $Q(\mathcal{O}) > 1$ such that

$$\sum_{i=1}^{M} \left| \frac{\partial f(u_0(y))}{\partial \xi_i} \right| \leq Q(\mathcal{O})$$

provided $y \in \mathcal{O}$. Let $\varepsilon_3 = \dfrac{\varepsilon}{16 \cdot K \cdot Q(\mathcal{O})}$ then for

$u \in W\left(\dfrac{\varepsilon}{32.K.Q(\sigma)}\right)$ we have

$$\| h'(x,y) - h(x,y) \|_M < \varepsilon_3 \ .$$

Let $\varepsilon_4 = \dfrac{\varepsilon}{8.M.K}$. Using again the uniform continuity of partial derivatives of the function f on σ , we obtain $\delta_4 > 0$ so that for each $u \in H^M_\alpha(\Omega)$, $u \in W(\delta_4)$, and for each $y \in \Omega$ it is

$$\left| \frac{\partial f(u(y))}{\partial \xi_i} - \frac{\partial f(u_0(y))}{\partial \xi_i} \right| < \varepsilon_4 \ .$$

So

$$C(u,x,y) \leq$$

$$\leq \frac{|f(u(y)+h'(x,y))-f(u(y))|}{\|h'(x,y)\|_M} \left| \frac{\|h'(x,y)\|_M}{\|x-y\|_N^\alpha} - \frac{\|h(x,y)\|_M}{\|x-y\|_N^\alpha} \right| +$$

$$+ \frac{\|h(x,y)\|_M}{\|x-y\|_N^\alpha} \left| \frac{f(u(y)+h'(x,y))-f(u(y))}{\|h'(x,y)\|_M} - \right.$$

$$\left. - \frac{f(u_0(y)-h'(x,y))-f(u_0(y))}{\|h(x,y)\|_M} \right| \leq K(\sigma)\varepsilon_1 + K \cdot D(u,x,y) \ ,$$

where

$$D(u,x,y) = \left| \frac{f(u(y)+h'(x,y))-f(u(y))}{\|h'(x,y)\|_M} - \frac{f(u_0(y)+h(x,y))-f(u_0(y))}{\|h(x,y)\|_M} \right| .$$

The relations (4) imply

$$D(u,x,y) = 2\varepsilon_2 + \left| \frac{\sum_{i=1}^{M} \frac{\partial f(u(y))}{\partial \xi_i} h'_i(x,y)}{\|h'(x,y)\|_M} - \frac{\sum_{i=1}^{M} \frac{\partial f(u_0(y))}{\partial \xi_i} h'_i(x,y)}{\|h'(x,y)\|_M} \right. +$$

$$+\left.\frac{\sum_{i=1}^{M}\frac{\partial f(u_0(y))}{\partial \xi_i}h'_i(x,y)}{\|h'(x,y)\|_M}-\frac{\sum_{i=1}^{M}\frac{\partial f(u_0(y))}{\partial \xi_i}h_i(x,y)}{\|h(x,y)\|_M}\right|\leq$$

$$\leq 2\varepsilon_2+\sum_{i=1}^{M}\frac{|h_i(x,y)|}{\|h'(x,y)\|_M}\cdot\left|\frac{\partial f(u(y))}{\partial \xi_i}-\frac{\partial f(u_0(y))}{\partial \xi_i}\right|+$$

$$+\sum_{i=1}^{M}\left|\frac{\partial f(u_0(y))}{\partial \xi_i}\right|\cdot\left|h'_i(x,y)\|h(x,y)\|_M-h_i(x,y)\|h'(x,y)\|_M\right|\leq$$

$$\leq 2\varepsilon_2+M\varepsilon_4+\sum_{i=1}^{M}\left|\frac{\partial f(u_0(y))}{\partial \xi_i}\right|\cdot|h'_i(x,y)|\cdot\left|\|h(x,y)\|_M-\|h'(x,y)\|_M\right|+$$

$$+\sum_{i=1}^{M}\left|\frac{\partial f(u_0(y))}{\partial \xi_i}\right|\cdot\|h'(x,y)\|_M\cdot|h'_i(x,y)-h_i(x,y)|\leq$$

$$\leq 2\varepsilon_2+M\varepsilon_4+2Q(\sigma)\varepsilon_3\ .$$

It means that

$$(5)\quad C(u,x,y)\leq K(\sigma)\varepsilon_1+2K\varepsilon_2+K\cdot M\cdot\varepsilon_4+$$
$$+2KQ(\sigma)\varepsilon_3=\frac{\varepsilon}{2}$$

provided $u\in W(\Delta)\cap W\left(\frac{\varepsilon}{32\cdot Q(\sigma)K}\right)\cap W(\delta_4)$ and

$x,y\in\Omega$, $x\neq y$, $\|x-y\|_N^{\infty}<\frac{\delta_2}{K}$

Now, let us suppose that $\|x-y\|_N^{\infty}\geq\frac{\delta_2}{K}$.

Then

$$(6)\quad\frac{|f(u(x))-f(u_0(x))-(f(u(y))-f(u_0(y))|}{\|x-y\|_N^{\infty}}\leq$$

$$\le \frac{2\mathcal{K}.\mathcal{K}(\sigma)}{\delta_2} \| u - u_0 \|_{\delta} \le \frac{\varepsilon}{2} \quad \text{for} \quad u \in W\left(\frac{\varepsilon \delta_2}{4.\mathcal{K}.\mathcal{K}(\sigma)}\right) .$$

From (3), (5) and (6) we have: for a given $\varepsilon > 0$ there exists $\delta > 0$ such that if $u \in W(\Delta) \cap W(\delta) \cap$

$\cap\ W\left(\frac{\varepsilon}{32\,\mathcal{K}.Q(\sigma)}\right) \cap W\left(\frac{\varepsilon\delta_2}{4\mathcal{K}.\mathcal{K}(\sigma)}\right)$ then

$$\| f \circ u - f \circ u_0 \|_{H_\alpha} < \varepsilon$$

which is nothing else than the continuity of $\mathcal{N}$.

Now, let us suppose that the operator $\mathcal{N}$ is continuous. Let $\xi_0 \in \mathbb{R}^M$, $y_0 \in \Omega$ be fixed. Define

$$u_0(x) = \| x - y_0 \|_N^\alpha \cdot j_1 + \xi_0 , \quad x \in \Omega ,$$

$$u_h(x) = \| x - y_0 \|_N^\alpha \cdot j_1 + \xi_0 + h ,$$

where $h \in \mathbb{R}^M$, $x \in \Omega$ and $j_1 = [1,0,\dots,0] \in \mathbb{R}^M$. From the continuity of $\mathcal{N}$ we have: For a given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{| f(u_h(x)) - f(u_h(y)) - (f(u_0(x)) - f(u_0(y)) |}{\| x - y \|_N^\alpha} < \varepsilon$$

provided $h \in \mathbb{R}^M$, $\| h \|_M < \delta$. Putting $y = y_0$ we obtain

$$(\gamma) \qquad \left| \frac{f(\xi_0 + \|x - y_0\|_N^\alpha \cdot j_1 + h) - f(\xi_0 + h)}{\|x - y\|_N^\alpha} - \right.$$
$$\left. - \frac{f(\xi_0 + \|x - y_0\|_N^\alpha j_1) - f(\xi_0)}{\|x - y_0\|_N^\alpha} \right| < \varepsilon$$

for all $x \in \Omega$.

Let $p = \{x \in \mathbb{R}^M \mid x = [t, 0, \ldots, 0] + \xi_0 ,\ t \in \mathbb{R}\}$.

The restriction of the function f on the straight line p is absolutely continuous, for it is locally lipschitzian on p (see Theorem 1). Thus the partial derivative $\frac{\partial f}{\partial \xi_1}$ exists and it is finite almost everywhere. Let us suppose, now, that there exists $\xi_0 \in p$ such that $\frac{\partial f}{\partial \xi_1}(\xi_0)$ does not exist, i.e.,

$$\overline{L} = \overline{\lim_{t \to 0}} \frac{f(\xi_0 + t j_1) - f(\xi_0)}{t} >$$
$$> \underline{\lim_{t \to 0}} \frac{f(\xi_0 + t j_1) - f(\xi_0)}{t} = \underline{L} .$$

Denote by $\{\tau_n\}$, $\{t_n\}$ the sequences of real numbers with the following properties:

$$\lim_{n \to \infty} \tau_n = 0 , \quad \lim_{n \to \infty} t_n = 0 ,$$

and

$$\overline{L} = \lim_{n \to \infty} \frac{f(\xi_0 + t_n j_1) - f(\xi_0)}{t_n}$$

$$\underline{L} = \lim_{n \to \infty} \frac{f(\xi_0 + \tau_n j_1) - f(\xi_0)}{\tau_n}$$

Let $x_n, y_n \in \Omega$ be such that

$$\| x_n - y_0 \|_N^{\alpha} = t_n \,, \quad \| y_n - y_0 \|_N^{\alpha} = \tau_m \,.$$

Moreover, let $\{h_n\}$ be such a sequence that $h_m \in \mathbb{R}$ with $\lim_{m\to\infty} h_m = 0$ and $\dfrac{\partial f(\xi_0 + h_m j_1)}{\partial \xi_1}$ exists for all $m \in \mathbb{N}$. Denote $\bar{\varepsilon} = \dfrac{\bar{L} - \underline{L}}{2}$ and, substituting in (7) $h = h_m j_1$, $x = x_n$, $\varepsilon = \bar{\varepsilon}$ or $x = y_n$, $h = h_m j_1$, $\varepsilon = \bar{\varepsilon}$, we return with

$$(8) \qquad \left| \frac{f(\xi_0 + t_m j_1 + h_m j_1) - f(\xi_0 + h_m j_1)}{t_m} - \frac{f(\xi_0 + t_m j_1) - f(\xi_0)}{t_m} \right| < \bar{\varepsilon}$$

$$(9) \qquad \left| \frac{f(\xi_0 + \tau_m j_1 + h_m j_1) - f(\xi_0 + h_m j_1)}{\tau_m} - \frac{f(\xi_0 + \tau_m j_1) - f(\xi_0)}{\tau_m} \right| < \bar{\varepsilon}$$

for sufficiently large $m, n \in \mathbb{N}$. Setting $n \to \infty$ we obtain from (8), (9)

$$(8') \qquad \left| \frac{\partial f(\xi_0 + h_m j_1)}{\partial \xi_1} - \bar{L} \right| \le \bar{\varepsilon} \,,$$

$$\left| \frac{\partial f(\xi_0 + h_m j_1)}{\partial \xi_1} - \underline{L} \right| \le \bar{\varepsilon} \,.$$

So the inequalities

$$2\bar{\varepsilon} < |\underline{L} - \bar{L}| \le \left| \frac{\partial f(\xi_0 + h_m j_1)}{\partial \xi_1} - \underline{L} \right| + \left| \frac{\partial f(\xi_0 + h_m j_1)}{\partial \xi_1} - \bar{L} \right| \le 2\bar{\varepsilon}$$

are valid for sufficiently large m . This is a contradiction.

Thus $\dfrac{\partial f}{\partial \xi_1}(\xi_0)$ exists for arbitrary $\xi_0 \in \mathbb{R}^M$.
The continuity of $\dfrac{\partial f}{\partial \xi_1}$ follows easily from (7), letting x tend to y_0 .

The proof of the existence and continuity of $\dfrac{\partial f}{\partial \xi_1}, \dots$
$\dots \dfrac{\partial f}{\partial \xi_M}$ is analogous.

4. Remarks.

A. Let us consider $0 < \beta \le \alpha \le 1$. Then the operator $\mathcal{N}$ maps $H^M_\alpha(\Omega)$ into $H_\beta(\Omega)$ if and only if f is such a function that for each bounded nonempty subset σ of $\mathbb{R}^M$ there exists $K(\sigma) > 0$ such that

$$|f(\xi) - f(\eta)| \le K(\sigma) \|\xi - \eta\|_M^{\frac{\beta}{\alpha}}$$

for each $\xi, \eta \in \sigma$. The proof is analogous to that of Theorem 1.

It seems that the necessary and sufficient conditions upon f for $\mathcal{N}$ to be continuous remain to be an open problem.

B. If $0 < \alpha < \beta \le 1$ then $\mathcal{N}$ maps $H^M_\alpha(\Omega)$ into $H_\beta(\Omega)$ if and only if f is a constant function: Let us suppose $\xi_0 \in \mathbb{R}^M$. Denote

$$u(x) = \| x - y_0 \|_M^{\alpha} \cdot j_1 + \xi_0 , \qquad x \in \Omega$$

(as in the proof of Theorem 2). Obviously $u \in H_{\alpha}^M(\Omega)$ and thus

$$\sup_{\substack{x \neq y_0 \\ x \in \Omega}} \frac{|f(u(x)) - f(u(y_0))|}{\|x - y_0\|^{\beta}} = L < +\infty .$$

Denoting $t = \| x - y_0 \|_N^{\alpha}$ we obtain

$$L t^{\frac{\beta}{\alpha}-1} \geq \left| \frac{f(\xi_0 + t j_1) - f(\xi_0)}{t} \right|$$

and so the right hand side derivative of $t \mapsto f(\xi_0 + t j_1)$ at $t = 0$ is zero. Similarly the same is valid for the left hand side derivative.

Thus $\frac{\partial f}{\partial \xi_1}(\xi_0) = 0$ and analogously $\frac{\partial f}{\partial \xi_2}(\xi_0) = \dots = \frac{\partial f}{\partial \xi_M}(\xi_0) = 0$.

C. The investigation of the same problems as in Theorems 1 and 2 for the operator

$$u(x) \longmapsto f(x, u(x)) ,$$

where $f(x, \xi)$ is a given function on $\Omega \times \mathbb{R}^M$, was without success.

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