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A PRODUCT INTEGRAL REPRESENTATION OF THE GENERA[illegible] INVERSE

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Abstract: Suppose $A \in \mathcal{L}(H_1, H_2)$ has closed range where H_1 and H_2 are Hilbert spaces. Let $S \subset [0, \infty)$ be such that $0 \in S$ and $\sup\{t: t \in S\} = \infty$ and suppose $g: S \to [0, \infty)$ is an increasing function with $g(0) = 0$ and $g(t) \to \infty$ as $t \to \infty$. Let $W(t) = {}_0\Pi^t [I - dg \tilde{A}]$ where $\tilde{A}$ is the restriction of $A^* A$ to $H = R(A^*)$ and suppose $W(t) \to 0$ uniformly in $\mathcal{L}(H,H)$ as $t \to \infty$. If $M(t) = (L) \int_0^t W(\cdot) A^* \, dg$ then $A^+ = \lim_{t \to \infty} M(t)$ uniformly in $\mathcal{L}(H_2, H_1)$. This generalizes some well known representations of A^+.

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1. Introduction. The concept of a product integral (or continuous product) of a matrix-valued function was introduced by Volterra [14] as a tool in the study of linear time-dependent differential equations. The theory was later extended and generalized by Schlesinger [12], Rasch [11] and Masani [9]. More recently McNerney [6, 7] has given a very general treatment and Martin [8] has applied the theory in approximating solutions of linear operator equations.

It is the purpose of this note to give a representation of the generalized inverse of a bounded linear operator in terms of product integrals of operator-valued functions. This representation unifies and generalizes some well known representations of the generalized inverse.

2. Product integrals and the generalized inverse. Since we will be concerned with linear operators in Hilbert space we will restrict our discussion of product integrals to operator-valued functions in Hilbert space although the concept can be extended to much more general settings (see [6],[7], [9]). Suppose $S \subset [0, \infty)$ satisfies $0 \in S$ and $\sup\{t: t \in S\} = \infty$. Let $g: S \to [0, \infty)$ be a function satisfying

(2.1) $g(0) = 0$; $g(t) \leq g(s)$ for $t \leq s$ and

$\sup\{g(t): t \in S\} = \infty$.

If H is a Hilbert space, given a function $\phi : S \to H$ and $t \in S$, then

$$(L) \int_0^t \phi(\cdot)\, dg$$

will denote the limit in the sense of refinements of subdivisions of elements of H of the form

$$\sum_{k=1}^{n} \phi(x_{k-1}) [g(x_k) - g(x_{k-1})]$$

where $(x_k)_0^n$ is a subdivision of $[0,t]$, i.e. $x_k \in S$, $x_0 = 0$, $x_n = t$ and $x_{k-1} \leq x_k$. If T is a bounded linear operator on H , i.e. $T \in \mathcal{L}(H,H)$ and $t \in S$ then

$$W(t) = {}_0\Pi^t [\, I - dgT \,]$$

denotes the member of $\mathcal{L}(H,H)$ which is the limit in the sense of refinements of subdivisions of operators of the form

$$\prod_{k=1}^{m} [\, I - (g(x_k) - g(x_{k-1}))T \,]$$

where $(x_k)_0^n$ is a subdivision of $[0,t]$ and I is the identity operator on H . By a result of MacNerney [7] (see also [8]) $W(t)$ is well-defined and satisfies

(2.2) $$W(t) = I - (L) \int_0^t W(\cdot)T \, dg \quad .$$

Now suppose that H_1 and H_2 are Hilbert spaces over the same scalars and $A \in \mathcal{L}(H_1,H_2)$ has closed range. Given $f \in H_2$, an element $u \in H_1$ is called a <u>least squares</u> solution of the equation

(2.3) $$Ax = f$$

if $\| Au - f \| = \inf \{ \| Ax - f \| : x \in H_1 \}$.

The set of least squares solutions of (2.3) coincides with the set of solutions of

(2.4) $$A^* Ax = A^* f$$

(where A^* is the adjoint of A) and there is a unique least squares solutions of smallest norm. The operator $A^+ \in \mathcal{L}(H_2,H_1)$ which assigns to each $f \in H_2$ the solution of (2.4) with smallest norm is called the generalized inverse of A . The set of least squares solutions of (2.3) can be represented as $A^+f \oplus \mathcal{N}(A)$ where $\mathcal{N}(A)$ is the nullspace of A . For more information and references see the survey article of Nashed [10].

Let $\tilde{A}$ be the operator defined on the Hilbert space $H = R(A^*)$ by restricting A^*A , i.e. $\tilde{A} = A^* A|_H$ (note that $R(A^*)$ is complete since A has closed range). Given a function g satisfying (2.1) and $t \in S$ define the operator $W(t) \in \mathcal{L}(H,H)$ by

$$W(t) = {}_0\Pi^t [I - dg\,\tilde{A}] .$$

Note that W(t) exists and is of bounded variation on each subinterval by results of MacNerney [6]. Also the operator $M(t) \in \mathcal{L}(H_2,H_1)$ defined by

$$M(t) = (L) \int_0^t W(\cdot)A^* dg$$

exists [7, Lemma 4.3] (see also [5, Lemma 2]).

<u>Theorem</u>. Suppose $A \in \mathcal{L}(H_1,H_2)$ has closed range and $\lim_{t\to\infty} W(t) = 0$ uniformly in $\mathcal{L}(H,H)$, then $A^+ = \lim_{t\to\infty} M(t)$ uniformly in $\mathcal{L}(H_2,H_1)$.

<u>Proof</u>. For each $f \in H_2$ note that $A^+f \in H$ (see [10])

and $\tilde{A}\ A^+ f = A^* f$. Hence we have by (2.2)

$$M(t)f = (L) \int_0^t W(\cdot)A^* f \ dg$$

$$= (L) \int_0^t W(\cdot)\tilde{A}\ A^+ f \ dg$$

$$= A^+f - W(t)A^+ f$$

and hence $M(t)f \longrightarrow A^+ f$ uniformly in $\mathcal{L}(H_2,H_1)$ as $t \longrightarrow \infty$ since $W(t) \longrightarrow 0$ uniformly in $\mathcal{L}(H,H)$.

Corollary 1. Suppose $0 < \lambda_k < 2\ \| A \|^{-2}$ and $\sum_{k=1}^{\infty} (1 - C_k) = \infty$ where $C_k = | 1 - \lambda_k \| A \|^2 |$, then

$$A^+ = \sum_{k=0}^{\infty} \lambda_{k+1} \{ \prod_{j=1}^{k} [I - \lambda_j \tilde{A}] \} A^*$$

where the convergence is in the uniform operator topology for $\mathcal{L}(H_2,H_1)$.

Proof. Set $S = \{0,1,2,\dots\}$ and $g(n) = \sum_{i=1}^{n} \lambda_i$. Note that the hypotheses imply that $\sum_{i=1}^{\infty} \lambda_i = \infty$. It is easy to see by use of the spectral theorem and standard facts on the convergence of infinite numerical products [3] that

$$W(n) = {}_1\prod^n [I - dg\ \tilde{A}] = \prod_{j=1}^{n} [I - \lambda_j \tilde{A}]$$

converges to 0 uniformly in $\mathcal{L}(H,H)$. The proof is completed by noting that

$$M(n) = (L) \int_0^n W(\cdot)\, A^* \, dg$$

$$= \sum_{k=0}^{n} \lambda_{k+1} \{ \prod_{j=1}^{k} [I - \lambda_j \tilde{A}] \} A^* .$$

Note that if we define a sequence of operators $\{A_n\}_0^{\infty} \subset \mathcal{L}(H_2,H_1)$ iteratively by $A_0 = 0$ and

$$A_{n+1} = A_n + \lambda_{n+1} [A^* - A^* A A_n]$$

then $A_{n+1} = M(n)$ and hence Corollary 1 gives an iterative scheme for computing A^+. Iterative processes of this type have recently been studied by Lardy [4] (see also [2]). In the particular case $\lambda_n = \lambda$ for all n where $0 < \lambda < 2 \|A\|^{-2}$ Corollary 1 specializes to give a result of Showalter [13]. We may also obtain as a corollary Showalter's integral representation of A^+ (see also [1]).

<u>Corollary 2</u>. Let $A_t^+ = \int_0^t e^{-\tau A^* A} A^* \, d\tau$, then $A^+ = \lim_{t \to \infty} A_t^+$ uniformly in $\mathcal{L}(H_2,H_1)$.

<u>Proof</u>. Here we take $S = [0, \infty)$ and $g(t) = t$. Then $W(t) = {}_0\prod^t [I - dg\, \tilde{A}] = e^{-t\tilde{A}}$ ([9],[12]) and hence $A^+ = \lim_{t \to \infty} \int_0^t e^{-\tau A^* A} A^* \, d\tau$.

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