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Jan Menu; Aleš Pultr

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ON CATEGORIES DETERMINED BY POSET- AND SET-VALUED FUNCTORS

Jan MENU, Antwerpen, and Aleš PULTR, Praha

Abstract: The present note is closely connected with the Manes characterization of lattice fiberings in [3]. On the one hand, in § 1 we give a characterization of a slightly more general case, namely that where the values of the inducing functor do not necessarily possess suprema of all subsets. On the other hand, § 2 deals with characterizing of two important particular cases of lattice fiberings, the categories $\mathcal{S}(F)$ (see below). The argument in § 1 is very close to that of Manes. What we show is that one can dispense of the assumption that the forgetful functor is (also) colimit preserving. In § 2, roughly speaking, the obvious necessary conditions are shown to be also sufficient.

Key words: Concrete category, lattice fiberings and their generalization, categories $\mathcal{S}(F)$.

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§ 0. Preliminaries

0.1. The category of all sets and mappings is denoted by **Set** , the category of partially ordered sets and order-preserving mappings is denoted by **Poset** . We use the symbol

$$\mathcal{D}$$

for the category of partially ordered sets such that every non-void subset has an infimum, and of the suprema preserving mappings.

0.2. A concrete category $(\mathfrak{K}, \mathrm{U})$ is a category together with a faithful functor $\mathrm{U} : \mathfrak{K} \longrightarrow \mathcal{S}et$ (called the forgetful functor). Concrete categories $(\mathfrak{K}, \mathrm{U})$ and $(\mathfrak{L}, V)$ are said to be equally carried if there is an isofunctor $\Phi : \mathfrak{K} \longrightarrow \mathfrak{L}$ such that $V \circ \Phi = \mathrm{U}$.

A concrete category $(\mathfrak{K}, \mathrm{U})$ is said to have the property of transfer (shortly,(T)) if for every $a \in \mathrm{obj}\, \mathfrak{K}$ and every invertible mapping $f : \mathrm{U}(a) \longrightarrow X$ there is an isomorphism $\varphi : a \longrightarrow b$ with $\mathrm{U}(\varphi) = f$ (cf. [1] and [4]).

0.3. Given a concrete category $(\mathfrak{K}, \mathrm{U})$ and a set X , we denote by

$$\mathfrak{K}\mathrm{U}X$$

the class of all $a \in \mathrm{obj}\, \mathfrak{K}$ with $\mathrm{U}(a) = X$, preordered by the relation

$$a \leq b \text{ iff there is a } \varphi : a \longrightarrow b \text{ such that } \mathrm{U}(\varphi) = 1_X .$$

Obviously, the transfer property implies that $\mathfrak{K}\mathrm{U}X$ and $\mathfrak{K}\mathrm{U}Y$ with equally large X and Y are **equivalent** .

0.4. Let H be a functor from $\mathcal{S}et$ into $\mathcal{P}oset$ or into $\mathcal{D}$. In accordance with the notation of [2] (where the symbol is used for the functors terminating in CSL , the full subcategory of $\mathcal{D}$ generated by the complete lattices) we denote by

$$\mathfrak{A}_H$$

the concrete category the objects of which are couples (X, a) with $a \in \mathrm{H}(X)$, the morphisms (X, a) into (Y, b) are triples (a, f, b) with $f : X \longrightarrow Y$ such that

$H(f)(a) \leq b$, the forgetful functor sending (a, f, b) to

0.5. Let F be a covariant (contravariant, resp.) functor from $\mathcal{S}et$ into itself. The concrete category

$$\mathcal{S}(F)$$

is defined as follows: The objects are couples (X, a) with $a \subset F(X)$, the morphisms from (X, a) into (Y, b) are triples (a, f, b) with $f: X \longrightarrow Y$ such that $F(f)(a) \subset$ $\subset b$ ($F(f)(b) \subset a$, resp.); the forgetful functor sends (a, f, b) to f .

The categories $\mathcal{S}(F)$ play a role in questions concerning description of morphisms. It was, e.g., proved in [1] that every reasonable concrete category is a full concrete subcategory of a suitable $\mathcal{S}(F)$.

The categories $\mathcal{S}(F)$ may be considered as a particular case of $\mathcal{A}_H$ defining $H(X) = (\exp F(X), \subset)$ and $H(f)(a) = F(f)(a)$ in the covariant, $H(X) = (\exp F(X), \supset)$ and $H(f)(a) = F(f)^{-1}(a)$ in the contravariant case.

0.6. The following trivial lemma will be of use in the both following paragraphs:

<u>Lemma</u>. Let $\mathcal{K}$ be complete (cocomplete, resp.), let $(\mathcal{K}, U)$ have (T) and let U preserve limits (colimits, resp.). Let $D: K \longrightarrow \mathcal{K}$ be a diagram, let $(f_k : Z \longrightarrow UD(k))_{k \in obj K}$ be a limit ($(f_k : UD(k) \longrightarrow Z)_{k \in obj K}$ a colimit, resp.) of $U \circ D$. Then there is a limit (colimit, resp.) $(\varphi_k)_{k \in obj K}$ of D such that $U(\varphi_k) = f_k$.

<u>Proof</u>: Take a limit $(\varphi'_k : x' \longrightarrow D(k))_k$ of D . Hence,

$(\mathrm{U}(\varphi'_{k}))_{k}$ is a limit of $\mathrm{U}D$, so that there is an invertible g with $\mathrm{U}(\varphi'_{k})\circ g = f_{k}$. By (T) we have an isomorphism $\gamma: x \longrightarrow x'$ such that $\mathrm{U}(\gamma) = g$. Put $\varphi_{k} = \varphi'_{k} \circ \gamma$.

§ 1. The categories $\mathcal{O}\!\ell_{H}$ with H terminating in $\mathcal{D}$ or Poset.

1.1. We introduce two further conditions on concrete categories:

($\mathcal{D}$) : Every $\mathcal{R}\mathrm{U}X$ is in $obj\,\mathcal{D}$.

(inf) : Let there be $\varphi_i : b \longrightarrow a_i$, $i \in \mathcal{J} \neq \emptyset$, such that $\mathrm{U}(\varphi_i) = f$ and let $\inf a_i$ exist. Then there is a $\varphi: b \longrightarrow \inf a_i$ with $\mathrm{U}(\varphi) = f$.

1.2. Remark. These conditions are not artificial. Obviously, they are satisfied in every complete concrete $(\mathcal{R}, \mathrm{U})$ such that U is a both-sided adjoint and every $\mathcal{R}\mathrm{U}X$ is a set (in that case, $\mathcal{R}\mathrm{U}X$ are complete lattices). Moreover, it is easy to show that they hold in reflective subcategories of such $(\mathcal{R}, \mathrm{U})$ if e.g. the reflection morphisms are extremal epimorphisms.

1.3. Theorem. Let $(\mathcal{R}, \mathrm{U})$ be a complete cocomplete concrete category with a limit preserving U. Let the condition ($\mathcal{D}$) be satisfied. Then the following three statements are equivalent:

(i) $(\mathcal{R}, \mathrm{U})$ has (T), and (inf) and for every $f: \mathrm{U}(a) \longrightarrow Y$ there is a $\varphi: a \longrightarrow b$ with $\mathrm{U}(\varphi) = f$.

(ii) $(\mathcal{R}, \mathcal{U})$ is equally carried with an $\mathcal{O}_H$ with $H : \mathbf{Set} \to \mathcal{D}$.

(iii) $(\mathcal{R}, \mathcal{U})$ is equally carried with an $\mathcal{O}_H$ with $H : \mathbf{Set} \to \mathbf{Poset}$.

1.4. Remark. For example of a category satisfying all the properties and such that $\mathcal{R}\mathcal{U}X$ have no non-trivial suprema consider the following one: The objects are couples (X, A) with $A \subset X$ and $\operatorname{card} A \leq 1$, the morphisms $(X,A) \to (Y,B)$ are the $f : X \to Y$ with $f(A) \subset B$.

1.5. Proof of 1.3:

(i) $\Longrightarrow$ (ii): For a set X put $H(X) = \mathcal{R}\mathcal{U}X$, for a mapping $f : X \to Y$ and an $a \in \mathcal{R}\mathcal{U}X$ put $H(f)(a) = \inf\{b \mid \exists \varphi : a \to b,\ \mathcal{U}(\varphi) = f\}$ (which exists by $(\mathcal{D})$ and the last condition in (i)). By (inf), we have a $\varphi : a \to H(f)(a)$ such that $\mathcal{U}(\varphi) = f$. Obviously,

$$a \leq b \quad \text{implies} \quad H(f)(a) \leq H(f)(b) .$$

Let $f : X \to Y$, $g : Y \to Z$ be mappings. For an $a \in H(X)$ we have a $\varphi : a \to H(f)(a)$ and a $\psi : H(f)(a) \to H(g)H(f)(a)$ such that $\mathcal{U}(\varphi) = f$ and $\mathcal{U}(\psi) = g$. Thus, $\mathcal{U}(\psi\varphi) = gf$ and hence

$$H(gf)(a) \leq H(g)H(f)(a) .$$

Take the $\iota : H(gf)(a) \to H(g)H(f)(a)$ with $\mathcal{U}(\iota) = 1_Z$ and the $\chi : a \to H(gf)(a)$ with $\mathcal{U}(\chi) = gf$. Since $\mathcal{U}$ is faithful, we have

$$\iota \circ \chi = \psi \circ \varphi .$$

By 0.6 there is a pullback

$$\begin{tikzcd} H(f)(a) \arrow[r, "\psi"] & H(g)H(f)(a) \\ c \arrow[u, "\iota'"] \arrow[r, "\psi'"] & H(gf)(a) \arrow[u, "\iota"'] \end{tikzcd}$$

such that $U(\iota') = 1_Y$. Hence, we have a λ with

$$\iota'\lambda = \varphi \quad \text{and} \quad \psi'\lambda = \chi .$$

By the definition of $H(f)(a)$, $\iota' = 1_{H(f)(a)}$. Thus, $\iota \circ \psi' = \psi$, and hence, by the definition of $H(g)H(f)(a)$, $\iota = 1$. Thus,

$$H(gf) = H(g)H(f) .$$

Now, we are going to show that $H(f)$ preserves suprema. Let a be the supremum of $\{a_i\}_{i \in J}$ in $\mathfrak{K}UX$. Put

$$b_i = H(f)(a_i) , \ b = H(f)(a) .$$

Thus, we have morphisms

$$\nu_i : a_i \longrightarrow a , \ \mu_i : b_i \longrightarrow b \quad \text{with } U(\nu_i) = 1_X , U(\mu_i) = 1_Y ,$$

$$\varphi_i : a_i \to b_i , \ \varphi : a \longrightarrow b \quad \text{with } U(\varphi_i) = U(\varphi) = f .$$

By ($\mathcal{D}$), any set with an upper bound has a supremum. Thus, there is a supremum c of $\{b_i\}$ and we have

$$\gamma_i : b_i \longrightarrow c \quad \text{with } U(\gamma_i) = 1_Y ,$$

$$\mu : c \longrightarrow b \quad \text{with } U(\mu) = 1_Y .$$

Hence, $\mu_i = \mu \circ \gamma_i$

Consider the colimit $(\nu_i' : a_i \longrightarrow a')_{i \in J}$ of the diagram consisting of all the a_i and all the identity carried morphism between them. If $U(\xi) = 1_X$ for a $\xi : a_i \longrightarrow a_j$, we have $U(\nu_j \xi) = 1 = U(\nu_i)$ so that $\nu_j \xi = \nu_i$. Similarly, $\gamma_j \varphi_j \xi = \gamma_i \varphi_i$. Thus, there are morphisms

$$\alpha : a' \longrightarrow a \quad \text{with} \quad \alpha \nu_i' = \nu_i \ ,$$

$$\psi : a' \longrightarrow c \quad \text{with} \quad \psi \nu_i' = \gamma_i \varphi_i \ .$$

Consider the diagram

$$\begin{tikzcd}
a \arrow[rrr, "\varphi"] & & & b \\
& a' \arrow[ul, "\alpha"'] \arrow[r, "\psi"] & c \arrow[ur, "\mu"] & \\
a_i \arrow[uu, "\nu_i"] \arrow[ur, "\nu_i'"'] \arrow[rrr, "\varphi_i"] & & & b_i \arrow[uu, "\mu_i"'] \arrow[ul, "\gamma_i"']
\end{tikzcd}$$

We have

$$\mu \psi = \varphi \alpha$$

(really, $\mu \psi \nu_i' = \mu \gamma_i \varphi_i = \mu_i \varphi_i = \varphi \nu_i = \varphi \alpha \nu_i'$),

and

$$H(U(\alpha))(a') = a$$

(really, let us have a $\beta : a' \longrightarrow \tilde{a}$ with $U(\beta) = U(\alpha)$; we have $U(\beta \nu_i') = U(\alpha) U(\nu_i') = U(\nu_i) = 1$ so that $\tilde{a} \geq \sup a_i = a$).

Consequently,

$$HU(\psi)(a') = H(U(\mu)U(\psi))(a') = H(U(\varphi)U(\alpha))(a') = H(f)(a) = b \ ,$$

so that, since $\psi : a' \longrightarrow c$, $b \leq c$, and hence $b = c$.

Thus, H is a functor from $\mathbf{Set}$ into $\mathcal{D}$.

Now, define functors

$$\Phi : \mathcal{K} \longrightarrow \mathcal{A}_H , \quad \Psi : \mathcal{A}_H \longrightarrow \mathcal{K}$$

putting

$\Phi(a) = (\mathrm{U}(a), a)$ and $\Phi(\varphi) = (a, \mathrm{U}(\varphi), b)$ for $\varphi : a \longrightarrow b$,

$\Psi(X, a) = a$ and $\Psi(a, f, b) = \varphi : a \longrightarrow b$ such that $\mathrm{U}(\varphi) = f$.

(The last definition is correct: there is at most one such φ by the faithfulness of U, and there exists one since there is a $\psi : a \longrightarrow H(f)(a)$ with $\mathrm{U}(\psi) = f$ and we have $H(f)(a) \leq b$.)

Obviously, $V \circ \Phi = \mathrm{U}$ for the natural forgetful functor V of $\mathcal{A}_H$.

Immediately by the definitions we see that $\Psi\Phi(\varphi) = \varphi$ and $\Phi\Psi(a, f, b) = (a, f, b)$, so that Φ, Ψ are isofunctors. Thus, (ii) is proved.

(ii)$\Longrightarrow$(iii): trivially.

(iii)$\Longrightarrow$(i): The property (T) is obvious. If (a, f, b) are morphisms, we have $H(f)(a) \leq b_i$ for all i, and hence $H(f)(a) \leq \inf b_i$, so that $(a, f, \inf b_i)$ is a morphism. Finally, $(a, f, H(f)(a))$ is always a morphism.

§ 2. The categories S(F).

2.1. Let us recall some well-known definitions and facts. If $\mathcal{L}$ is a lattice and o its least (e its largest, resp.) element, an element $a \in \mathcal{L}$ is said to be an atom (a coatom, resp.) of $\mathcal{L}$ if

$$a \neq \sigma \quad \text{and } (\bigvee_J x_i \geq a \Longrightarrow \exists i,\ x_i \geq a)$$

$$(a \neq e \quad \text{and } (\bigwedge_J x_i \leq a \Longrightarrow \exists i,\ x_i \leq a, \text{ resp.}) .$$

The set of all atoms (coatoms, resp.) of $\mathcal{L}$ will be denoted by

$$\mathcal{A}(\mathcal{L}) \ (\mathcal{C}(\mathcal{L}), \text{ resp.}) .$$

In a Boolean algebra, the atoms coincide with the minimal elements, i.e. with the a such that

$$a \neq \sigma \quad \text{and } (x \leq a \ \& \ x \neq \sigma \Longrightarrow x = a) .$$

A lattice $\mathcal{L}$ is said to be atomic (coatomic, resp.) if

$$\forall x \in \mathcal{L} \quad x = \bigvee \{a \mid a \leq x \ \& \ a \in \mathcal{A}(\mathcal{L})\}$$

$$(\forall x \in \mathcal{L} \quad x = \bigwedge \{a \mid a \geq x \ \& \ a \in \mathcal{C}(\mathcal{L})\}, \text{ resp.}).$$

If $\mathcal{L}$ is a Boolean algebra, then a is an atom iff $\neg a$ is a coatom.

A Boolean algebra is atomic iff it is coatomic.

For an atomic Boolean algebra $\mathcal{L}$, the formula

$$\iota(x) = \{a \mid a \leq x \ \& \ a \in \mathcal{A}(\mathcal{L})\}$$

defines an isomorphism of $\mathcal{L}$ onto $(\exp \mathcal{A}(\mathcal{L}), \subset)$.

2.2. <u>Theorem</u>. A concrete category $(\mathcal{K}, \mathcal{U})$ is equally carried with an $S(F)$ with a covariant F iff the following conditions are satisfied:

(i) $\mathcal{K}$ is cocomplete and $\mathcal{U}$ preserves colimits.

(ii) $(\mathcal{K}, \mathcal{U})$ has (T) .

(iii) Every $\mathcal{K}\mathcal{U}X$ is a set and an atomic Boolean algebra.

(iv) Denote by σ_X the zero of $\mathcal{R}\mathfrak{U}X$. For every $f : X \to Y$ there is a $\varphi : \sigma_X \to \sigma_Y$ with $\mathfrak{U}(\varphi) = f$.

(v) If there is a $\psi : a \to \sigma_X$ then $a = \sigma_{\mathfrak{U}(a)}$.

(vi) For every $f : X \to Y$ and for every $a \in \mathfrak{a}(\mathcal{R}\mathfrak{U}X)$ there is a $\varphi : a \to b$ with $\mathfrak{U}(\varphi) = f$ and $b \in \mathfrak{a}(\mathcal{R}\mathfrak{U}Y)$.

Remark. Obviously, by (iv), $\mathfrak{U}$ is a right adjoint. (Moreover, it has a left adjoint $\mathbf{L}$ such that $\mathfrak{U} \circ \mathbf{L} = 1_{Set}$.)

Proof: For an $S(F)$, the conditions (i) - (vi) are obviously satisfied. Now, let the conditions hold. First, we will prove the following statement:

If $a \in \mathfrak{a}(\mathcal{R}\mathfrak{U}X)$, $b \in \mathfrak{a}(\mathcal{R}\mathfrak{U}Y)$ and if $\varphi : a \to b$ is a morphism then

$$(*) \qquad \begin{array}{ccc} \sigma_X & \xrightarrow{\ \psi\ } & \sigma_Y \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota'} \\ a & \xrightarrow{\ \varphi\ } & b \end{array}$$

with $\mathfrak{U}(\psi) = \mathfrak{U}(\varphi)$, $\mathfrak{U}(\iota) = 1_X$ and $\mathfrak{U}(\iota') = 1_Y$ is a pushout.

Really,

$$\begin{array}{ccc} X & \xrightarrow{\ \mathfrak{U}(\psi) = \mathfrak{U}(\varphi)\ } & Y \\ \downarrow{\scriptstyle 1_X = \mathfrak{U}(\iota)} & & \downarrow{\scriptstyle 1_Y = \mathfrak{U}(\iota')} \\ X & \xrightarrow{\ \mathfrak{U}(\varphi)\ } & Y \end{array}$$

is a pushout and hence by (i), (ii) and 0.6 there is a push-out

$$\begin{tikzcd} \sigma_X \arrow[r, "\psi"] \arrow[d, "\iota"] & \sigma_Y \arrow[d, "\iota''"] \\ a \arrow[r, "\varphi'"] & c \end{tikzcd}$$

with $\mathfrak{U}(\iota'') = 1_Y$ and $\mathfrak{U}(\varphi') = \mathfrak{U}(\varphi)$. Thus, there is a $\varkappa : c \longrightarrow b$ such that $\varkappa\iota'' = \iota'$. Hence, $\mathfrak{U}(\varkappa) = 1$ and therefore $c \leq b$. Thus, either $c = b$, $\iota' = \iota''$ and $\varphi' = \varphi$, or $c = \sigma_Y$. The second alternative is, however, excluded by (v).

Now, define

$$F : \mathbf{Set} \longrightarrow \mathbf{Set}$$

putting $F(X) = \mathcal{A}(\mathcal{R}\mathfrak{U}X)$ and, for $f : X \longrightarrow Y$ and $a \in \mathcal{A}(\mathcal{R}\mathfrak{U}X)$, $F(f)(a) = b \in \mathcal{A}(\mathcal{R}\mathfrak{U}Y)$ such that there is a $\varphi : a \longrightarrow b$ with $\mathfrak{U}(\varphi) = f$. Such a b exists by (vi) and it is uniquely determined by ($*$). Obviously, F is a covariant functor.

For an $x \in obj\,\mathcal{R}$ put $\kappa(x) = \{a \mid a \in F\mathfrak{U}(x)\ \&\ a \leq x\}$ and define

$$\Phi : \mathcal{R} \longrightarrow S(F) \quad \text{and} \quad \Psi : S(F) \longrightarrow \mathcal{R}$$

by

$$\Phi(x) = (\mathfrak{U}(x), \kappa(x)), \quad \Phi(\varphi) = (\kappa(x), \mathfrak{U}(\varphi), \kappa(y))$$

for $\varphi : x \longrightarrow y$,

$$\Psi(X, \kappa) = \bigvee\{a \mid a \in \kappa\}, \quad \Psi(\kappa, f, s) = \varphi : \Psi(X, \kappa) \longrightarrow \Psi(Y, s)$$

such that $\mathfrak{U}(\varphi) = f$.

The definitions are correct: If $a \in \kappa(x)$, we have a $\lambda : a \longrightarrow x$ with $\mathfrak{U}(\lambda) = 1_{\mathfrak{U}(x)}$. By ($*$) we have the pushout

$$\begin{array}{ccc} \sigma_{\mathfrak{U}(x)} & \xrightarrow{\varphi'} & \sigma_{\mathfrak{U}(y)} \\ \downarrow \iota & & \downarrow \iota' \\ a & \xrightarrow{\varphi''} & F\mathfrak{U}(\varphi)(a) \end{array}$$

with $\mathfrak{U}(\varphi') = \mathfrak{U}(\varphi'') = \mathfrak{U}(\varphi)$, and we have a $\varkappa : \sigma_{\mathfrak{U}(y)} \longrightarrow y$ with $\mathfrak{U}(\varkappa) = 1$. Since $\mathfrak{U}$ is faithful, we have $\varkappa \varphi' = \varphi \iota$. Thus, there is a $\varkappa'$ with $\varkappa' \iota' = \varkappa$, and hence $F\mathfrak{U}(\varphi)(a) \leq y$, i.e. $F\mathfrak{U}(\varphi)(a) \in \varkappa(y)$. There is at most one φ satisfying the formula for $\Psi(\varkappa, f, \rho)$ so that it suffices to prove its existence. It is, however, easy to check (using 0.6) that $\Psi(X, \varkappa)$ is a colimit of the diagram consisting of the σ_X , all the $a \in \varkappa$, and all the $\iota : \sigma_X \longrightarrow a$ with $\mathfrak{U}(\iota) = 1$, from which the existence of φ immediately follows.
Now, since $\mathfrak{K}\mathfrak{U}X$ are atomic, we have $\Psi\Phi(x) = x$ and $\Psi\Phi(\varphi) = \varphi$. Since it is a Boolean algebra, $\Phi\Psi(X, \varkappa) = (X, \varkappa)$ and hence obviously $\Phi\Psi(\varkappa, f, \rho) = (\varkappa, f, \rho)$. Thus Φ and Ψ are isofunctors and obviously $V \circ \Phi = \mathfrak{U}$ where V is the forgetful functor of $S(F)$.

2.3. By a quite analogous reasoning one obtains the following

<u>Theorem</u>. A concrete category $(\mathfrak{K}, \mathfrak{U})$ is equally carried with an $S(F)$ with a contravariant F iff the following conditions are satisfied:

(i') $\mathfrak{K}$ is complete and $\mathfrak{U}$ preserves limits.

(ii) $(\mathfrak{K}, \mathfrak{U})$ has (T) .

(iii) Every $\mathfrak{K}\mathfrak{U}X$ is a set and an atomic Boolean algebra.

(iv′) Denote by e_X the unit of $\mathcal{R}\mathfrak{U}X$. For every $f : X \longrightarrow Y$ there is a $\varphi : e_X \longrightarrow e_Y$ with $\mathfrak{U}(\varphi) = f$.

(v′) If there is a $\psi : e_X \longrightarrow a$ then $a = e_{\mathfrak{U}(a)}$.

(vi′) For every $f : X \longrightarrow Y$ and for every $b \in \mathcal{C}(\mathcal{R}\mathfrak{U}Y)$ there is a $\varphi : a \longrightarrow b$ with $\mathfrak{U}(\varphi) = f$ and $a \in \mathcal{C}(\mathcal{R}\mathfrak{U}X)$.

2.4. Remarks. The first four conditions from 2.2 are, of course, satisfied also in the contravariant case 2.3, and vice versa.
The remaining two conditions, however, are characteristic for the variance. In fact, the only case when the both collections are satisfied is the one of $S(F)$ with F a constant (and hence both co- and contravariant) functor.

Really, suppose that an $S(F)$ with a covariant F satisfies (i′) - (vi′). Since the coatoms are the objects $(X, F(X) \setminus \{u\})$ with $u \in F(X)$, we obtain by (vi′) that for every $f : X \longrightarrow Y$ and every $v \in F(Y)$ there is a $u \in F(X)$ such that $F(f)(F(X) \setminus \{u\}) \subset F(Y) \setminus \{v\}$. Consequently,

every $F(f)$ is one-to-one.

Denote by γ_X the unique mapping $X \longrightarrow P$ where P is a fixed one-point set. If X is non-void, we have a δ_X with $\gamma_X \delta_X = 1_P$, and hence $F(\gamma_X)$ is onto. Since also $F(\gamma_\emptyset)$ is onto by (v′), we see that

every $F(\gamma_X)$ is invertible.

Put $A = F(P)$ and consider the constant functor C_A defined by $C_A(f) = 1_A$. Put $\varepsilon^X = F(\gamma_X)$. Since we have, for any $f : X \longrightarrow Y$, $\gamma_Y \circ f = \gamma_X$, we obtain

$$\varepsilon^Y \circ F(f) = F(\gamma_Y f) = \varepsilon^X = \varepsilon^X \circ C_A(f) \; .$$

Thus, ε is a natural equivalence $F \cong C_A$.

R e f e r e n c e s

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Rijksuniversitair Centrum Antwerpen
Dienst voor Wiskundige Analyse
2020 - Antwerpen,Middleheimlaan 1
België

Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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