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Václav Koubek

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EACH CONCRETE CATEGORY HAS A REPRESENTATION BY T_2 PARACOMPACT TOPOLOGICAL SPACES

Václav KOUBEK, Praha

Abstract: It is shown that every concrete category can be fully embedded into a category whose objects are paracompact Hausdorff spaces and whose morphisms are all non-constant continuous (or closed continuous) mappings **between** these spaces.

Key words: Concrete category, full embedding, paracompact Hausdorff space, continuous mapping, closed continuous mapping.

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The aim of the paper is to prove that each concrete category is isomorphic to a category whose objects are paracompact connected Hausdorff spaces and whose morphisms are all non-constant continuous (closed continuous, respectively) mappings between these objects. The theorem is based on the fact that each concrete category is fully embeddable into $\mathcal{S}(P_2)$ proved in [3] by Kučera.

A similar result was obtained by V. Trnková [5] who proved an analogical theorem for metric (or compact Hausdorff) spaces under the assumption of the non-existence proper class of measurable cardinals. The present results do not require any special set-theoretical assumption.

The author would like to express his gratitude to V. Trnková who introduced him to this problematics.

Convention: Denote $P_A = \langle - , A \rangle$ the contravariant hom-functor from the category of all sets and their mappings into itself.

Definition. Let F be a contravariant functor from sets to sets. Denote $S(F)$ the category, objects of which are couples $(X, \mathcal{U})$, X being a set, $\mathcal{U} \subset FX$, and $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is a morphism if $f : X \longrightarrow Y$ is a mapping with $Ff(\mathcal{V}) \subset \mathcal{U}$. In particular, objects of $S(P_2)$ are couples $(X, \mathcal{U})$, $\mathcal{U} \subset \exp X$ and morphisms $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ are mappings such that $f^{-1}(A) \in \mathcal{U}$ for each $A \in \mathcal{V}$.

Theorem 1. Every concrete category can be fully embedded into the category $S(P_2)$.

Proof: see [3].

Theorem 2. There exists a metric continuum M such that if Z is a subcontinuum of M, $f : Z \longrightarrow M$ is a continuous mapping then either f is constant or $f(x) = x$ for all $x \in Z$. M has $\aleph_0$ pairwise disjoint subcontinua.

Proof: see [1].

Convention: For a given topological space T, T^X denote, the topological product of topological spaces T_i, $i \in X$, where each T_i is homeomorphic to T. Let T_i, $i \in I$ be topological spaces, then $\bigvee_{i \in I} T_i$ denote, the topological sum of topological spaces T_i, $i \in I$.

Convention: Denote Z the set of all integers. Choose arbitrary but fixed disjoint subcontinua A, B, C_z, $z \in Z$ of M. Notice that the only continuous mappings between these three spaces are constants and the identities of A, B, C_z, $z \in Z$.

Theorem 3. There exists a full embedding $\Phi : S(P_2) \longrightarrow S(P_A)$.

Proof: see [4].

Definition. A topological space T is stiff if every continuous mapping $f : T \longrightarrow T$ is either the identity or a constant.

Theorem 4. Let T be a stiff Hausdorff space. Let $f : T^Q \longrightarrow T$ be a continuous mapping. Then f is either a projection or a constant.

Proof: see [2].

Corollary 5: Let T be a stiff Hausdorff space. Then $f : T^Q \longrightarrow T^R$ is a continuous mapping if and only if there exists a partial mapping $g : R \longrightarrow Q$ and a point $a \in T^R$, $a = \{a_i\}_{i \in R}$, such that for every $x \in T^Q$, $f(x) = y = \{y_i\}_{i \in R}$ where $y_i = x_{g(i)}$ if $g(i)$ is defined, $y_i = a_i$ otherwise.
In particular, $f : T \longrightarrow T^N$ is a continuous mapping if and only if there exists $N' \subset N$ and $a = \{a_i\}_{i \in N} \in T^N$ such that $f(x) = y = \{y_i\}_{i \in N}$ and $y_i = x$ if $i \in N'$, $y_i = a_i$ otherwise.

Corollary 6: The only continuous mappings between A^N and either B or C_z, $z \in Z$, are constants.

Lemma 7. Let K be a subcontinuum of a Hausdorff space Q, let $a, b \in K$, $a \neq b$ such that $M = K - \{a, b\}$ is open in Q. Then for each continuous mapping $f : Z \longrightarrow Q$, where Z is a continuum, either there exists a component H of $f^{-1}(K)$ such that $a, b \in f(H)$ or there exists a continuous mapping $\tilde{f} : Z \longrightarrow Q$ such that $\tilde{f} = f$ on $f^{-1}(Q - M)$ and $\tilde{f}(f^{-1}(K)) \subset \{a, b\}$.

Proof: see [5].

Construction 8: In each C_z, $z \in Z$, choose a pair distinct points c_z, d_z. Define a topological space $D = \bigvee_{z \in Z} C_z / \sim$, where $d_z \sim c_{z+1}$ for every $z \in Z$.
Choose distinct points $a_1, a_2 \in A$, $b_1, b_2 \in B$. For given set X define a topological space $E_X = A^X \vee (B \times \{0,1\}) / \approx$, where $\{0,1\}$ is a discrete topological space and $a' = \{a'_x\}_{x \in X} \approx \{b_1, 0\}$, $\{b_2, 0\} \approx \{b_1, 1\}$, $\{b_2, 1\} \approx \bar{a} = \{\bar{a}_x\}_{x \in X}$, where $a'_x = a_1$, $\bar{a}_x = a_2$ for every $x \in X$.
For each object $P = (X, \mathcal{U})$ of $S(P_A)$ denote by P^* the space $E_X \vee (D \times \mathcal{U})$, where $\mathcal{U}$ is the discrete topological space with underlying set $\mathcal{U}$. Let $\tilde{P}$ be a coarser topological space than P^* : a set V, open in P^* is open in $\tilde{P}$ if and only if for each $u \in \mathcal{U} \subset A^X$ either $u \notin V$ or there exists n_0 with $\bigcup_{n > n_0} C_n \times \{u\} \subset V$ and either $\{b_2, 0\} \notin V$ or there exists n_1 with $\bigcup_{n < n_1} C_n \times \mathcal{U} \subset V$; clearly $\tilde{P}$ is a connected paracompact Hausdorff space.
Define a contravariant functor ψ from $S(P_A)$ into the

category PAR of connected paracompact Hausdorff spaces: $\psi P = \widetilde{P}$, $\psi f = (P_A f \vee (1_B \times \{0,1\}))/\approx \vee (1_D \times P_A f/\mathcal{U})/\sim$,

where 1_B and 1_D are the identities of B and D . Clearly, ψf is correctly defined and it is a closed continuous mapping.

Evidently the functor ψ is faithful.

<u>Lemma 9</u>. Let $f: T \longrightarrow \widetilde{P}$ be a non-constant continuous mapping.

a) If $T = A$ then $f(T) \subset A^X$;

b) if $T = B$ then $f(T) \subset B \times \{i\}$, where $i \in \{0,1\}$.

c) If $T = C_z$ then $f(T) \subset D \times \{u\}$ for some $u \in \mathcal{U}$.

In all above cases, f is an embedding.

Proof: Let K, a, b denote one of the following:

a) $K = C_z \times \{u\}$, $a = \langle c_z, u \rangle$, $b = \langle d_z, u \rangle$ for some $z \in Z$, $u \in \mathcal{U}$.

b) $K = B \times \{i\}$, $a = \langle b_1, i \rangle$, $b = \langle b_2, i \rangle$ for some $i \in \{0,1\}$.

Suppose that the former case in Lemma 7 takes place, i.e. that there is a component L of $f^{-1}(K)$ with $a, b \in f(L)$. Then we get easily by Theorem 2 that L is homeomorphic to T and f is a homeomorphism of T onto K . Now, suppose that, for all K, a, b as above, the latter case in Lemma 7 takes place.

1) Suppose that $f(T)$ meets the interior of some K , where K is from a). Then apply Lemma 7 on f, $K' = C_{z-1} \times \{u\}$, $\langle c_{z-1}, u \rangle$, $\langle d_{z-1}, u \rangle$ to obtain $\widetilde{f}$

and again Lemma 7 to $\tilde{f}$, $K'' = C_{z+1} \times \{u\}$, $\langle c_{z+1}, u \rangle$, $\langle d_{z+1}, u \rangle$ to obtain $\tilde{\tilde{f}}$. Then $\tilde{\tilde{f}}$ coincides with f on $f^{-1}(K)$ and $\tilde{\tilde{f}}(T)$ is a continuum which does not meet the interiors of both K' and K'' but it meets the interior of K . Then, as easily seen from the construction of $\tilde{P}$, $\tilde{\tilde{f}}(T) \subset K$. By Theorem 2, $\tilde{\tilde{f}}$ is an embedding of T onto K and $f = \tilde{\tilde{f}}$.

2) Let the assumption of 1) not hold. Then $f(T) \subset A^X \cup \cup B \times \{0, 1\}$ as for any continuum which does not meet the interior of any K from a)..

Let us apply Lemma 7 on f, $B \times \{0\}$, $\langle b_1, 0 \rangle$, $\langle b_2, 0 \rangle$ to obtain $\tilde{f}$ and again Lemma 7 on $\tilde{f}$, $B \times \{1\}$, $\langle b_1, 1 \rangle$, $\langle b_2, 1 \rangle$ to obtain $\tilde{\tilde{f}}$.

If $\tilde{f}$ is constant then clearly $f(T) \subset B \times \{0\}$ and f is an embedding by Theorem 2. Analogously, if $\tilde{\tilde{f}}$ is constant then $\tilde{f}$ is an embedding of T onto $B \times \{1\}$ and so is f . Let $\tilde{\tilde{f}}$ be non-constant. As $\tilde{\tilde{f}}(T) \subset A^X$, we may apply Corollaries 5, 6. We obtain that $\tilde{\tilde{f}}$ is an embedding of T into A^X and so is f .

Lemma 10. Let $f : \tilde{P} \longrightarrow \tilde{R}$ be a continuous mapping $P, R \in S(P_A)$ with $f/B \times \{0\} = 1_{B \times \{0\}}$ .Then there exists $\varphi : R \longrightarrow P$ such that $\psi \varphi = f$.

Proof: Lemma 9 implies either $f/B \times \{1\} = 1_{B \times \{1\}}$ or $f(B \times \{1\}) = \langle b_1, 1 \rangle$. If $f(B \times \{1\}) = \langle b_1, 1 \rangle$ then $f(\bar{a}) = \langle b_1, 1 \rangle$ and therefore there exists $h : A \longrightarrow \tilde{R}$ such that $\langle b_1, 0 \rangle, \langle b_2, 0 \rangle \in h(A)$ but this is impossible. Hence $f/B \times \{1\} = 1_{B \times \{1\}}$. Denote Δ_X the diagonal of A^X , Δ_Y the diagonal of A^Y , where $P = (X, \mathcal{U})$,

$R = (Y, \mathcal{V})$. We have $f(\Delta_X) = \Delta_Y$ and so $f(A^X) \subset A^Y$. Corollary 5 implies that there exists $q : Y \longrightarrow X$ such that $f/A^X = P_A q$. As $f(\langle b_1, 1\rangle) = \langle b_1, 1\rangle$ and $f(A^X) \subset A^Y$, $f/D \times \{u\}$ is an embedding from $D \times \{u\}$ into $D \times \{f(u)\}$ and therefore $f/D \times \mathcal{U} = 1_D \times P_A q/\mathcal{U}$ and $P_A q(\mathcal{U}) \subset \subset \mathcal{V}$. Hence $\psi q = f$.

<u>Lemma 11</u>. Let $f : \tilde{P} \longrightarrow \tilde{R}$ be a continuous mapping such that $f/B \times \{0\} \neq 1_{B \times \{0\}}$. Then f is constant.

Proof: Assume that $f/B \times \{0\}$ is non-constant. Then Lemma 9 implies that $f/B \times \{0\}$ is an embedding and so $f(\langle x, 0\rangle) = \langle x, 1\rangle$ for every $x \in B$. Therefore $f(\langle b_1, 1\rangle) = f(\langle b_2, 0\rangle) = \langle b_2, 1\rangle$ and by Lemma 9 we have $f(B \times \{1\}) = \langle b_2, 1\rangle$. Hence $\langle b_2, 1\rangle \in f(\Delta_X)$ and $\langle b_2, 0\rangle \in f(\Delta_X)$ which is a contradiction (see Lemma 9). Therefore $f/B \times \{0\}$ is constant by Lemma 9. Analogously $f/B \times \{1\}$ is constant and so is f/Δ_X. Therefore f/A^X is constant by Lemma 9 and so is f.

<u>Definition</u>. Let $\mathcal{K}, \mathcal{L}$ be concrete categories. A functor $\mathcal{D} : \mathcal{K} \longrightarrow \mathcal{L}$ is an almost full embedding of $\mathcal{K}$ into $\mathcal{L}$ if $\mathcal{D}$ is an embedding of $\mathcal{K}$ onto a subcategory of $\mathcal{L}$ whose objects are $\mathcal{D}(a)$, a running over objects of $\mathcal{K}$ and whose morphisms are all non-constant $\mathcal{L}$-morphisms between these objects.

<u>Theorem 12.</u> Denote **PAR** the category of paracompact connected Hausdorff spaces and continuous mappings, $\mathbf{PAR}_c$ its subcategory with the same objects and continuous closed mappings as morphisms. Then each category $\mathbf{L}$

with

$$\mathrm{PAR}_c \subset L \subset \mathrm{PAR}$$

is almost universal in the sense that each concrete category has an almost full embedding into L.

Theorem 12 follows from Construction 9 and Lemmas 10 and 11.

A class C of topological spaces is called <u>stiff</u> for every continuous mapping $f: T \longrightarrow T'$, with $T, T' \in C$, is either constant or the identity of the space $T = T'$ onto itself.

V. Trnková had constructed a stiff class (= not a set) of paracompact spaces as follows.

Let H_i, $i = 1, \dots, 5$ be five disjoint subcontinua of the Cook continuum. Choose points $a, b, p_2, p_3 \in H_1$, $r_i, s_i \in H_i$, $i = 2, \dots, 5$, all distinct. For each ordinal α and $i = 1, \dots, 5$, put $H_i^\alpha = \{(x, \alpha) \mid x \in H_i\}$, $\varphi_i^\alpha(x, \alpha) = x$. We write x^α instead of (x, α). Let ω be an ordinal. Put

$$Q_\omega = (\bigcup_{\alpha \in \omega} H_1^\alpha \setminus \{b^\alpha\}) \cup (\bigcup_{\substack{i = 2,3 \\ \alpha \in \omega}} H_i^\alpha \setminus \{r_i^\alpha, s_i^\alpha\}) \cup$$
$$\cup (H_4^0 \setminus \{r_4^0, s_4^0\}) \cup H_5^\omega) .$$

$G \subset Q_\omega$ is open iff it fulfils (1) - (5).

(1) $\varphi_i^\alpha(G \cap H_i^\alpha)$ is open in H_i for all $i = 1, \dots, 5$, $\alpha \leq \omega$;

(2) if $\alpha \in \omega$, $a^\alpha \in G$ then

$\varphi_4^0(G \cap H_4^0)$ is a nbh of r_4 in H^4 whenever $\alpha = 0$

$\varphi_1 (G \cap H_1^{\beta})$ is a *nbh* of b_1 in H_1 whenever $\alpha = \beta + 1$

G contains H_1^{γ} for all $\alpha' \leq \gamma < \alpha$ (and some $\alpha' < \alpha$) whenever α is limit;

(3) if $\alpha \in \omega$, $i = 2, 3$, $p_i^{\alpha} \in G$, then $\varphi_i^{\alpha} (G \cap H_i^{\alpha})$ contains a *nbh* of x_i in H_i ;

(4) if $x_5^{\omega} \in G$ then G contains H_1^{γ} for all $\alpha' \leq \gamma < \omega$ (and some $\alpha' < \omega$).

(5) if $s_5^{\omega} \in G$, then $\varphi_i^{\alpha} (G \cap H_i^{\alpha})$ contains a *nbh* of s_i in H_i for all $(i, \alpha) = (0, 4), (\omega, 5)$ or $i = 2, 3$ and $\alpha \in \omega$.

By means of Lemma 7, one can prove that $\{Q_{\omega} \mid 1 \leq \alpha\}$ is a stiff proper class of paracompact spaces.

The existence of a stiff proper class of paracompact spaces follows also from the main result because "large discrete category" can be almost fully embedded in PAR .

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Matematicko-fyzikální fakulta
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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