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WHEN A GENERALIZED ALGEBRAIC CATEGORY IS MONADIC

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Abstract: A necessary and sufficient condition for set functors F and G is given in the paper so that a generalized algebraic category $A(F,G)$ is monadic, i.e. it has a free algebra over any set.

Key words: Set functor, free algebra, monad, left adjoint, universal algebra.

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Given functors F, G from sets to sets, form a generalized algebraic category $A(F,G)$ (see [1],[2],[6],[7],[8]): objects are pairs (X,ω), where X is a set and ω maps FX into GX; morphisms $f:(X,\omega)\longrightarrow(X',\omega')$ are mappings $f: X\longrightarrow X'$ such that the diagram

$$\begin{array}{ccc} FX & \longrightarrow & GX \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\ FX' & \longrightarrow & GX' \end{array}$$

commutes. This notion generalizes the categories of universal algebras of a given type Δ (this is the case G equals identical functor, F is the sum of hom-functors $Hom(\alpha,-)$, where α are the cardinals in Δ).

Another generalization of universal algebras is represented by algebras over a monad. The present paper is devoted to the study of the interrelation of these two generalizations. We give a necessary and sufficient condition on F and G for the natural forgetful functor $A(F, G) \longrightarrow \mathcal{Set}$ to be monadic: The functor G is representable and F is not excessive (i.e., roughly speaking, F does not increase powers of arbitrary big sets).

Generalized algebraic categories were defined by Trnková and Goralčík in connection with Wyler's paper. They were investigated in a number of papers. We are much indebted to V. Trnková who introduced us to the topics.

I.

We work in the Gödel-Bernays set theory; the class of all ordinal numbers is denoted by $\mathcal{On}$, its subclass of all cardinal numbers is denoted by $\mathcal{Cn}$. $|X|$ denotes the cardinality of a set X, and if $f: X \longrightarrow Y$ is a mapping then $\mathcal{Im}\, f = \{f(x); x \in X\}$. By ordinal m we mean a set of all ordinals less than m. If m is a cardinal then m^+ is the cardinal succesor of m, $\mathrm{cf}\, m$ is a cofinal of m, i.e. the least cardinal n such that $m = \bigcup_{i<n} m_i$, where $m_i < m$.

A category of all sets and their mappings we denote $\mathcal{Set}$. By a set functor we shall mean a covariant functor from $\mathcal{Set}$ into itself. We denote shortly

$Q_M = Hom(M,-): Set \longrightarrow Set$ for every set M . Further $C_{\emptyset,1}: Set \longrightarrow Set$ is a set functor which is defined by $C_{\emptyset,1}\emptyset = \emptyset$, $C_{\emptyset,1}X = 1$, where $X \neq \emptyset$; $C_M: Set \longrightarrow Set$ is a constant functor to a set M . Let F, G be set functors. If F is naturally equivalent with G, then we shall write $F \simeq G$.

In our investigation of set functors we are deeply utilizing properties of "filters" of points of a set functor F , i.e. $\mathcal{F}_F^X(x) = \{Y \subset X;\ x \in Im\ Fi,\ i: Y \longrightarrow X$ is an inclusion$\}$ where X is a set and $x \in FX$.

We recall some facts about them from [3],[4],[6]:

A cardinal $\alpha > 1$ is said to be an unattainable cardinal of a set functor F if $F\alpha \neq \bigcup_{X<\alpha} \bigcup_{f: X \to \alpha} Im\ Ff$. We denote $\|\mathcal{F}_F^X(x)\| = min\{|Y|;\ Y \in \mathcal{F}_F^X(x)\}$.

Lemma I.1 [6]: Let F be a set functor, X a set $x \in FX$. If $\alpha = \|\mathcal{F}_F^X(x)\| > 1$ then $\alpha \geq \aleph_0$ is an unattainable cardinal of F .

Lemma I.2 [4]: Let F be a set functor, α be an unattainable cardinal of F . Then $|F\alpha| > \alpha$.

Let F be a set functor, $f: X \longrightarrow Y$ a mapping, $x \in FX$. We are using this notation: $f(\mathcal{F}_F^X(x)) = \{Y' \subset Y;\ \exists X' \in \mathcal{F}_F^X(x),\ Y' \supset f(X')\}$.

Lemma I.3 [3]: Let F be a set functor, $f: X \longrightarrow Y$ a mapping, $x \in FX$ then $f(\mathcal{F}_F^X(x)) \subset \mathcal{F}_F^Y(Ff(x))$ and if there exists $Z \in \mathcal{F}_F^X(x)$ with f/Z being one-to-one then $f(\mathcal{F}_F^X(x)) = \mathcal{F}_F^Y(Ff(x))$.

<u>Proposition I.4</u> : Let F be a set functor, α a singular cardinal and let there exist some $x \in F\alpha$ such that $sup\{z ; z \in Z\} = \alpha$ for every $Z \in \mathcal{F}_F^\alpha(x)$. Then $|F\alpha| > \alpha$.

<u>Proof</u>. Let $\{\sigma_i ; i \in cf\,\alpha\}$ be an increasing sequence of cardinals with $sup\{\sigma_i ; i \in cf\,\alpha\} = \alpha$. Let $\mathcal{M}$ be a maximal subset of $\{f ; f: cf\,\alpha \longrightarrow \alpha\}$ fulfilling the following conditions:

a) $\sigma_i \le f(i) < \sigma_i^+$ for every $i \in cf\,\alpha$ and every $f \in \mathcal{M}$;

b) $|Im\, f \cap Im\, g| < cf\,\alpha$ for every distinct mappings $f, g \in \mathcal{M}$.

We shall show that then $|\mathcal{M}| > \alpha$. Suppose that $|\mathcal{M}| \le \alpha$. Let $\psi : \alpha \longrightarrow \mathcal{M}$ be a surjection. Define a mapping $h : cf\,\alpha \longrightarrow \alpha$ by $h(0) = 0$, $h(i) = (sup\{\varphi(\gamma)(i); \gamma \in \sigma_i\}) + 1$ for $0 < i < cf\,\alpha$.

It is easy to verify that $\mathcal{M} \cup \{h\}$ fulfils the conditions a) and b). Thus a contradiction with the maximality of $\mathcal{M}$ is established.

Denote $\beta = min\{Z ; Z \in \mathcal{F}_F^\alpha(x)\}$, put $B = \alpha \times \beta$. For every $f \in \mathcal{M}$ we choose some $\varphi_f : \alpha \longrightarrow B$ such that $\varphi_f(i) \in \{f(j)\} \times \beta$ where $\sigma_j \le i < \sigma_{j+1}$ and there exists $Z \in \mathcal{F}_F^\alpha(x)$ with φ_f / Z is one-to-one. For every distinct mappings $f, g \in \mathcal{M}$ it holds $\varphi_f(\mathcal{F}_F^\alpha(x)) \neq \varphi_g(\mathcal{F}_F^\alpha(x))$ because $|Im\, f \cap Im\, g| < cf\,\alpha$

and for every $Z \in \mathcal{F}_F^{\alpha}(x)$ $\sup \{z ; z \in Z\} = \alpha$. Lemma I.3 implies $F\varphi_f(x) \neq F\varphi_g(x)$. Obviously, $|B| = \alpha$ and $|FB| > |B|$.

II.

Construction II.1: For any set functor F and arbitrary sets M and X we shall construct the transfinite sequence $\{W_\alpha(F, M, X), \alpha \in \mathcal{O}n\}$ by putting:

$W_0 = X \times \{0\}$

$W_1 = W_0 \cup (FW_0 \times M \times \{1\})$

$W_{\alpha+1} = W_\alpha \cup ((FW_\alpha - \bigcup_{\beta \in \alpha} FW_\beta) \times M \times \{\alpha + 1\})$

$W_\alpha = \bigcup_{\beta \in \alpha} W_\beta$ for every limit ordinal α .

We shall say that the sequence $\{W_\alpha(F, M, X), \alpha \in \mathcal{O}n\}$ stops if there exists some $\alpha \in \mathcal{O}n$ with $W_\alpha = W_{\alpha+1}$ i.e. $FW_\alpha = \bigcup_{\beta \in \alpha} FW_\beta$.

Proposition II.2: Let F be a set functor and M be an arbitrary non-empty set. Then for every set X such that $|FY| > |Y|$ whenever $|Y| \geq |X|$ the sequence $\{W_\alpha(F, M, X), \alpha \in \mathcal{O}n\}$ does not stop.
Proof follows immediately from the fact that for every $\alpha \in \mathcal{O}n$, $|W_{\alpha+1}| \geq |FW_\alpha| > |W_\alpha|$.

Proposition II.3: Let F be a set functor, M be a set. If for a set X there exists a cardinal β such that $\beta \geq |X \times M|$ and $|F\beta| \leq \beta$ then $\{W_\alpha(F, M, X), \alpha \in \mathcal{O}n\}$ stops and for every $\alpha \in \mathcal{O}n$ $|W_\alpha(F, M, X)| \leq \beta$.

Proof. It is easy to verify by transfinite induction that $|W_\alpha| \leq \beta$ for $\alpha < \beta$. We shall prove that $FW_\beta = \bigcup_{\delta \in \beta} FW_\delta$. By I.2 $|F\beta| = \beta$ guarantees that β is not an unattainable cardinal of F. Let $x \in FW_\beta$. Put $\varepsilon = \| \mathcal{F}_F^{W_\beta}(x) \|$ then by I.1 either $\varepsilon \leq 1$ or ε is an unattainable cardinal of F and therefore $\varepsilon < \beta$. If there exists $Z \in \mathcal{F}_F^{W_\beta}(x)$ such that $Z \subset W_\delta$ for some $\delta \in \beta$ then $x \in FW_\delta$ and thus $x \in W_\beta$. If for every $Z \in \mathcal{F}_F^{W_\beta}(x)$ and for every $\delta \in \beta$ we have $Z - W_\delta \neq \emptyset$ then there exists $\varphi : W_\beta \longrightarrow \beta \times \varepsilon$ such that φ / Z is a monomorphism for some $Z \in \mathcal{F}_F^{W_\beta}(x)$ and $\varphi(W_{\delta+1} - W_\delta) \subset \{\delta\} \times \varepsilon$ for every $\delta \in \beta$. Hence $\varphi(Z)$ is unbounded in lexicographic ordering of the set $\varepsilon \times \beta$ for every $Z \in \mathcal{F}_F^{W_\beta}(x)$. By I.4 we would have $|F\beta| > \beta$ which contradicts our assumption.

Let F, G be functors, X be a set. The object (Z, ω) of $A(F, G)$ shall be called a free algebra over X if $X \subset Z$ and for every $f : X \longrightarrow Y$ and every object (Y, τ) of $A(F, G)$ there exists the unique morphism $g : (Z, \omega) \longrightarrow (Y, \tau)$ with $g / X = f$.

Proposition II.4: Let G be a functor with $G \not\cong C_{\emptyset,1}$ and $G \not\cong Q_M$ for every M. Then $A(F, G)$ has no free algebra over any $X \neq \emptyset$ whenever $F \not\cong C_\emptyset$.

Proof. If G is not a factorfunctor of some Q_M then for every $x \in GX$ there exists $y \in GY$ with $Y \neq \emptyset$ and $Gf(x) \neq y$ for every $f : X \longrightarrow Y$. Therefore for every object (X, ω), $X \neq \emptyset$ there exists an object (Y, τ) such that $Y \neq \emptyset$ and there is no morphism $f : (X, \omega) \longrightarrow (Y, \tau)$. Let G be a factorfunctor of Q_M, $G \not\cong Q_N$ for any N and $G \not\cong C_{\emptyset,1}$. In this case we shall find for every set X, $X \neq \emptyset$, a pair formed by an object (Y, τ) and a mapping f, $f : X \longrightarrow Y$ with the following property: for every (Z, ω) with $X \subset Z$ there exist different mappings $g, h : (Z, \omega) \longrightarrow (Y, \tau)$ such that $g/X = h/X = f$. It means there is no free algebra over X .

a) If there exists $x_0 \in GY$ such that $exp\, Y \neq \mathcal{F}_G^Y(x_0)$ $Y_0 \notin \mathcal{F}_G^Y(x_0)$, where $V_0 = \bigcap \{V;\ V \in \mathcal{F}_G^Y(x_0)\}$ we can choose $f : X \longrightarrow Y$ such that $(Y - Im\, f) \in \mathcal{F}_G^Y(x_0)$. Let $\tau : FY \longrightarrow GY$ be constant mapping to x_0 . Now, if there exists a morphism $h : (Z\, \omega) \longrightarrow (Y, \tau)$ such that $X \subset Z$ and $h/X = f$ then $Im\, h \in \mathcal{F}_G^Y(x_0)$. Hence there exists a, $b \in Im\, h - Im\, f$ such that $Im\, h - \{a, b\} \in \mathcal{F}_G^Y(x_0)$. Consider a mapping $p : Y \longrightarrow Y$ such that $p(x) = x$ for $x \in Y - \{a, b\}$, $p(a) = b$, $p(b) = a$. Then $p \circ h \neq h$, $p \circ h/X = f$ and $p : (Y, \tau) \longrightarrow (Y, \tau)$ is a morphism of $A(F, G)$. Hence there exists no free algebra over X .

b) Suppose that there exists $x_0 \in GY$ such that $\emptyset \neq V_0 \in \mathcal{F}_G^Y(x_0)$, where $V_0 = \bigcap \{V;\ V \in \mathcal{F}_G^Y(x_0)\}$ and the transformation $\varepsilon : Q_{V_0} \longrightarrow G$, $\varepsilon^{V_0}(id_{V_0}) = x_0$ is not a monotransformation. Let $g : Y \longrightarrow V_0$ be a mapping such that $g \circ j = id_{V_0}$ where j is an inclusion from V_0 into Y . Choose a point v with $v \notin V_0$ and define $V_1 = V_0 \cup \{v\}$. Put $G(i \circ g)(x_0) = y_0$ where $i : V_0 \longrightarrow V_1$ is an inclusion. Let $\tau : FV_1 \longrightarrow GV_1$ be a constant mapping to y_0 and $f : X \longrightarrow V_1$ be a constant mapping to v . Then there exist distinct mappings $h, k : V_0 \longrightarrow U$ with $\varepsilon^U(h) = \varepsilon^U(k)$.

Hence $G h'(y_0) = G k'(y_0)$ where $h', k' : V_1 \longrightarrow U \vee \{v\}$ $h'/V_0 = h$, $k'/V_0 = k$, $h'(v) = k'(v) = v$. If for some (Z, ω), $Z \supset X$ there exists a morphism $\varphi : (Z, \omega) \longrightarrow$ $\longrightarrow (V_1, \tau)$ such that $\varphi/X = f$ then $h' \circ \varphi \neq k' \circ \varphi$ and $h' \circ \varphi/X = k' \circ \varphi/X = f \circ k'$. Further $k', h' : (V_1, \tau) \longrightarrow$ $\longrightarrow (U \vee \{a\}, \tau')$ are morphisms of $A(F, G)$ where τ' is constant to $G h'(y_0)$. Hence there exists no free algebra over X.

If G is a factorfunctor of some Q_M and $G \ncong C_{\emptyset,1}$ and $G \ncong Q_N$ for any N then it must hold either case a) or case b).

Proposition II.5 [1]: Let G be a set functor, $G \ncong Q_N$ for every N. Then $A(F, G)$ has a free algebra over $\emptyset$ if and only if $F\emptyset = \emptyset$.

Proposition II.6: $A(F, Q_M)$ has a free algebra over X if and only if $\{W_\alpha(F, M, X), \alpha \in On\}$ stops.

Proof. If there exists $\alpha \in On$ such that $W_\alpha = W_{\alpha+1}$ then put $Z = W_\alpha$. Define $\omega : FZ \longrightarrow Q_M Z$ such that $(\omega(x))(m) = \langle x, m, \delta_x + 1 \rangle$ where $x \in FZ$, $m \in M$ and $\delta_x = \min\{\delta \in \alpha ; x \in FW_\delta\}$. An easy verification that (Z, ω) is a free algebra over X is left to the reader. We assume that $\{W_\alpha(F, M, X), \alpha \in On\}$ does not stop. Then by II.3 $|FY| > |Y|$ for every $|Y| \geq |X|$. Denote by (A, ω) a free algebra of the category $A(F, Q_M)$ over X. Choose arbitrarily some $k \in W_0$. Define $\omega_\alpha : FW_\alpha \longrightarrow$ $\longrightarrow Q_M W_\alpha$ such that $(\omega_\alpha(x))(m) = \langle x, m, \delta_x + 1 \rangle$ for $x \in \bigcup_{\beta \in \alpha} FW_\beta$, where $\delta_x = \min\{\delta \in \alpha ; x \in FW_\delta\}$ and

$(\omega_\alpha(x))(m) = k$ for $x \in FW_\alpha - \bigcup_{\beta\in\alpha} FW_\beta$. Let $\varphi_\alpha : (A, \omega) \longrightarrow$ $\longrightarrow (W_\alpha, \omega_\alpha)$ be a morphism of the category $A(F, Q_M)$ such that $\varphi_\alpha / W_0 = j_\alpha : W_0 \longrightarrow W_\alpha$ is an inclusion. We shall prove by transfinite induction that $W_\beta \subset Im\, \varphi_\alpha$ for every $\alpha, \beta \in On$ with $\beta \in \alpha$. Evidently $W_0 \subset Im\, \varphi_\alpha$ for every $\alpha \in On$. Let $\beta \in \alpha$ and $W_\gamma \subset Im\, \varphi_\alpha$ for every $\gamma \in \beta$. If β is a limit ordinal, then $W_\beta = \bigcup_{\gamma\in\beta} W_\gamma \subset Im\, \varphi_\alpha$. If $\beta = \gamma + 1$, then $W_\beta = W_\gamma \cup ((FW_\gamma - \bigcup_{\delta\in\gamma} FW_\delta) \times M \times \{\beta\})$. Evidently $(Im\, \varphi_\alpha, \omega_\alpha / F(Im\, \varphi_\alpha))$ is a subalgebra of $(W_\alpha, \omega_\alpha)$, so $(\omega_\alpha(x))(M) \subset Im\, \varphi_\alpha$, i.e. $\langle x, m, \delta_x \rangle \in$ $\in Im\, \varphi_\alpha$ for every $m \in M$, $x \in FW_\gamma - \bigcup_{\delta\in\gamma} FW_\delta$. Since $|\alpha| \leq |Im\, \varphi_\alpha|$ it follows that $|A| \geq |\alpha|$ for every $\alpha \in$ $\in On$, which establishes a contradiction.

<u>Corollary II.7</u>: Let F be a set functor, M be a non-empty set. Then $A(F, Q_M)$ has a free algebra over X if and only if there exists a cardinal α such that $|F\alpha| \leq \alpha$ and $\alpha > |X \times M|$. If (Y, ω) is a free algebra over X in $A(F, Q_M)$ then $|Y| = min\{|Z|; |FZ| \leq |Z| \geq |X|\}$.

A set functor F is an excessive functor if there exists a cardinal α such that for every set Y, $|FY| > |Y|$ whenever $|Y| \geq \alpha$.

<u>Theorem II.8</u>: Let F, G be set functors, $F \not\simeq C_\emptyset$, $G \not\simeq C_{\emptyset,1}$, $G \not\simeq C_1$. Then the natural forgetful functor $\square : A(F, G) \longrightarrow Set$ has a left adjoint if and only if F is not excessive and $G \simeq Q_M$ for some M .
The theorem follows from the preceding propositions.

Proposition II.9: If $F \simeq C_{\emptyset}$ or $G \simeq C_1$ then the natural forgetful functor $\Box : A(F, G) \longrightarrow \mathcal{S}et$ has a left adjoint.

If $G \simeq C_{\emptyset,1}$ then the natural forgetful functor $\Box : A(F, G) \longrightarrow \mathcal{S}et$ has a left adjoint if and only if $F\emptyset = \emptyset$.

Proof is trivial.

Theorem II.10: Let F, G be set functors, $F \not\simeq C_{\emptyset}$, $G \not\simeq C_{\emptyset,1}$, $G \not\simeq C_1$. A forgetful functor $\Box : A(F, G) \longrightarrow \mathcal{S}et$ is monadic if and only if F is not excessive and $G \simeq Q_M$ for some M .

Proof. By a similar way as in [5] it can be proved that $\Box$ creates coequalizers for those parallel pairs f, g in $A(F, G)$ for which f, g has an absolute coequalizer in $\mathcal{S}et$. Thus, in virtue of the Beck's theorem (see[5]), $\Box$ is monadic whenever $\Box$ has a left adjoint.

Remark II.11: By a similar way as in II.3 it can be proved without the assumption of the generalized continuum hypothesis that every excessive functor fulfils the condition R_M from [6] for every set M . Thus all the results concerning sums in $A(F, G)$ from [6] and [1] remain valid without the assumption of the generalized continuum hypothesis, i.e. the following theorem holds:

Theorem: Let F, G be functors. Then $A(F, G)$ has sums if and only if

either F preserves sums

or F preserves unions, $|G1| = 1$ and G preserves collective monomorphisms

or F is not excessive and $G \simeq Q_M$ for some M

or $G \simeq C_{\emptyset,1}$ and $F\emptyset = \emptyset$ or $F \simeq C_\emptyset$ or $G \simeq C_1$.

R e f e r e n c e s

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