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A NOTE ON LINE GRAPHS

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Abstract: Let G be a graph such that no component of G is a tree. In this note, a relationship between spanning subgraphs of G and spanning subgraphs of the line graph of G is discussed.

Key words: Graph; line graph; subdivision graph; spanning subgraph; homeomorphism; contraction.

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If G is a graph, then we denote by $V(G), E(G), \delta(G)$, $L(G)$ and $S(G)$ the vertex set of G, the edge set of G, the minimum degree of G, the line graph of G and the subdivision graph of G, respectively. For the terms and symbols not defined here, see Behzad and Chartrand [1], or Harary [3]. In the present note, we shall prove the following theorem:

Theorem. Let G be a graph such that no component of G is a tree. Then for every spanning subgraph F of G with $\delta(F) \geq 1$, there exists a spanning subgraph H of $L(G)$ such that (i) H is homeomorphic with F, and (ii) if $F = G$, then $S(G)$ is contractible to H.

Proof. Denote $V = V(G)$. If $v \in V$, then we denote by $D(v)$ the set of edges of G incident with v. If $A \subset V$, then we denote

$$\mathcal{D}(A) = \bigcup_{v \in A} \mathcal{D}(v) \ .$$

Assume that there is $B \subset V$ such that $|\mathcal{D}(B)| < |B|$. We denote by G_B the subgraph of G induced by B. Obviously, G_B contains a component $\widetilde{G}$ such that $|\mathcal{D}(\widetilde{V})| < |\widetilde{V}|$, where $\widetilde{V} = V(\widetilde{G})$. This implies that $\widetilde{G}$ is a tree and $\mathcal{D}(\widetilde{V}) = E(\widetilde{G})$. Thus $\widetilde{G}$ is a component of G, which is contradiction.

We have that for every $A \subset V$, $|A| \leq |\mathcal{D}(A)|$. From P. Hall's Theorem ([2], see also Theorem 12.3 in [1] or Theorem 5.19 in [3]) it follows that for every $u \in V$, there exists an edge $g(u) \in \mathcal{D}(u)$ such that if $v, w \in V$, $v \neq w$, then $g(v) \neq g(w)$. Denote $X = \{g(u) \mid u \in V\}$. Let $x \in E(G)$, $x = rs$. If $x \in X$, then $x \in \{g(r), g(s)\}$. If $x \notin X$, then x is adjacent both to $g(r)$ and to $g(s)$

Let F be a spanning subgraph of G with $\delta(F) \geq 1$, We denote by F_0 the graph with $V(F_0) = X \cup E(F)$ and such that distinct vertices y and z of F_0 are adjacent in F_0 if and only if there are $u_0, v_0 \in V$ such that

$$u_0 v_0 \in (E(F) \cap \{y, z\} \quad \text{and} \quad y, z \in \{u_0 v_0, g(u_0), g(v_0)\}.$$

It is easily seen that if y_0 and z_0 are adjacent vertices of F_0, then they are adjacent edges of G. Thus F_0 is a subgraph of $L(G)$.

Denote $Y = E(F) - X$. Every $y \in Y$ is a vertex of degree 2 in F_0 and if $y_1, y_2 \in Y$, then y_1 and y_2 are not adjacent in F_0. Let $r_1, r_2 \in V$ and $r_1 r_2 \in E(F)$; then either (1) $g(r_1)$ and $g(r_2)$ are adjacent vertices of F_0 or (2) there exists $y' \in Y$ which is adjacent both to $q(x_1)$ and to $q(x_2)$ in F_0. Let x_1 and x_2 be adjacent vertices of F_0, $x_1 \in X$; then there are s_1, $s_2 \in V$ such that $s_1 s_2 \in E(F)$, $x_1 = q(s_1)$, and either $x_2 = q(s_2)$ or $x_2 = s_1 s_2$. This implies that F_0 is homeomorphic with F and that $S(F)$ is contractible to F_0. For $F = G$, the proof is complete.

Let $F \neq G$. Denote $Z = E(G) - V(F_0)$. For every $z_0 \in Z$, let $a(z_0)$ be one of the vertices x_0 and s_0, where $z_0 = x_0 s_0$. If $w_0 \in V$, then we denote $B(w_0) = \{z \in Z \mid a(z) = w_0\}$. Denote $V_1 = \{u \in V \mid deg_F u = 1, q(u) \in E(F)\}$.

Obviously, if $t \in V - V_1$, then there are $x_t, y_t \in V(F_0)$ such that $x_t y_t$ is an edge of F_0, and t is incident both with x_t and with y_t in G. We denote by F' the graph which we obtain from F_0 in such a way that for every $v \in V - V_1$, we insert precisely $|B(v)|$ new vertices of degree 2 into the edge $x_v y_v$ of F_0. Clearly, F' is homeomorphic with F. We denote by F'' the graph which we obtain from F' in such a way that for every $w \in V_1$, we insert precisely $|B(w)|$ new vertices of degree 2 into the edge of F' which is incident with $q(w)$. Clearly, F'' is homeomorphic with F. It is not difficult to see that F'' is isomorphic to a spanning subgraph of $L(G)$. Hence the theorem.

<u>Corollary</u> (J. Sedláček [4]). Let G be a nontrivial connected graph. If G contains a hamiltonian path, then $L(G)$ also contains a hamiltonian path. If G contains a hamiltonian cycle, then $L(G)$ also contains a hamiltonian

cycle.

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