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# REMARK ON PERIODIC SOLUTIONS OF A LINEAR WAVE EQUATION IN ONE DIMENSION

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Abstract: In the present remark we are going to prove certain improvements of results of the papers [3] and [4].

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We consider the problem

$$(1) \quad u_{tt} - u_{xx} = f(t,x), \quad t \in (-\infty, \infty) = R, \quad x \in (0,\pi)$$

$$u(t,0) = u(t,\pi) = 0, \quad t \in R,$$

where $f$ and $u$ are periodic in $t$ with the same period $\omega > 0$, $f$ is supposed to be measurable in $R \times (0,\pi)$. We recall (see [3], Theorem 1.1, p. 155) that our problem has the generalized solution (for the definition see [3], p. 152) if and only if the following conditions hold:

a) $f_{k\ell} = 0$ for every couple $k, \ell$ with $\ell^2 - k^2 \alpha^2 = 0$, 1)

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1) $k$ and $\ell$ mean integers, $\ell > 0$.

b)

$$(2) \qquad \sum_{k,l} \frac{f^2_{kl}}{(l^2 - k^2\alpha^2)} < +\infty ,$$

where the summation runs over all $k, l$, $l^2 - k^2\alpha^2 \neq 0$. Here, $f_{kl}$ are the Fourier coefficients of $f(t,x)$, i.e.

$$f(t,x) \approx \sum_{k,l} f_{kl} e^{i\alpha k t} \sin lx ,$$

$\alpha = \dfrac{2\pi}{\omega}$. By $H_m$ denote the set of all $f$, whose derivative (with respect to $t$) of the order $m$ is finite and of class $L^2(T)$, $T = (0,\omega) \times (0,\pi)$. In other words, $H_m$ is exactly the set of all $f$ with

$$(3) \qquad \sum_{k,l} f^2_{kl} |k|^{2m} < +\infty$$

Usually continued fractions technique now gives easily the following

<u>Theorem 1</u>. Let $\gamma = \gamma(\alpha)$ be the supremum of all $\beta$ for which the inequality $|q\alpha - n| < q^{-\beta}$ has infinitely many solutions in positive integers $q, n$. Let $m$ be a positive number. Then

a) for every $f$, $f \in H_m$ there is a generalized solution of (1) provided $\gamma < m + 1$;

b) there exists a function $f$ of class $H_m$ such that (1) does not admit the generalized solution provided

$\gamma > m + 1$.

<u>Proof</u>. a) Let $\gamma < m+1$. Then, for $0 < \varepsilon < m + 1 - \gamma$ and every $k$ and $\ell$ it is

$$|\ell^2 - k^2\alpha^2| = (|\ell| + |k|\alpha)|\,|\ell| - |k|\alpha| \geq c|k|^{1-\varepsilon-\gamma}$$

with a suitable positive constant $c = c(\varepsilon, \alpha)$, i.e.

$$|\ell^2 - k^2\alpha^2| \geq c|k|^{-m}$$

and thus (3) implies (2).

b) Let $m+1 < \gamma < +\infty$, $m+1 < \varepsilon + m + 1 < \gamma$. Let $p_n/q_n$ be the convergents of $\alpha$, i.e. if $(a_0; a_1, a_2, \dots)$ is the continued fraction expansion of $\alpha$, $q_{-1} = 0$, $p_{-1} = 1$, $p_0 = a_0$, $q_0 = 1$, then

$$p_{n+1} = a_n p_n + p_{n-1},$$

$$q_{n+1} = a_n q_n + q_{n-1}, \qquad n = 0, 1, 2, \dots .$$

It is well known (see [1]) that

(4) $$\frac{c_2}{q_{n+1}} < |q_n\alpha - p_n < \frac{c_1}{q_{n+1}},$$

(5) $$c_4 q_n < p_n < c_5 q_n,$$

(6) $$q_n \geq c_5^n,$$

(7) $$\limsup_{n\to+\infty} \frac{\lg q_{n+1}}{\lg q_n} = \gamma,$$

where $c_1, c_2, \dots, c_5$ are positive constants, $c_5 > 1$.
Put

$$f_{q_m, p_m} = q_m^{-m-\varepsilon}, \qquad f_{k\ell} = 0 \quad \text{otherwise.}$$

Thus

$$\sum_{k,\ell} f_{k\ell}^2 \, |k|^{2m} = \sum_{m=0}^{\infty} \frac{1}{q_m^{2\varepsilon}} < +\infty$$

(see (6) ) and the corresponding function $f(t,x)$ is of class $H_m$. Further, by (4) - (6)

$$|p_m^2 - q_m^2 \alpha^2| = (q_m \alpha + p_m) |q_m \alpha - p_m| \geq c_6 \, q_m \, q_{m+1}^{-1} ,$$

$$\frac{f_{q_m p_m}}{|p_m^2 - q_m^2 \alpha^2|} \geq c_7 \, \frac{q_{m+1}}{q_m^{m+1+\varepsilon}}$$

with suitable positive constants $c_6$ and $c_7$. Thus by (7)

$$\limsup_{m \to +\infty} \frac{f_{q_m p_m}}{|p_m^2 - q_m^2 \alpha^2|} = +\infty$$

and the condition (2) does not hold.

For $\gamma = +\infty$ we choose a sequence of indices $m_1 < m_2 < \dots$ such that

$$\lim_{i \to +\infty} \frac{q_{m_i+1}}{q_{m_i}^{i+1}} = +\infty$$

and put

$$f_{q_{m_i} p_{m_i}} = \frac{1}{q_{m_i}^{i}}, \quad f_{k\ell} = 0 \qquad \text{otherwise.}$$

Then

$$\lim_{i \to +\infty} \frac{f_{q_{m_i} p_{m_i}}}{|p_{m_i}^2 - q_{m_i}^2 \alpha^2|} = +\infty$$

and for every $m$

$$\sum_{k,l} f_{kl}^2 |k|^{2m} = \sum_{i=1}^{\infty} \frac{q_{n_i}^{2m}}{q_{m_i}^{2i}} < +\infty .$$

<u>Remark</u>. If $\gamma = m+1$ then the existence of the generalized solution depends on the choice of $\alpha$ with $\gamma(\alpha) = m+1$. Examples can be easily constructed by means of continued fractions.

O. Vejvoda in the paper [4] (see p.347) proved that the necessary and sufficient condition for the existence of classical solution of Problem (1) is the existence of the function $s(x)$ of class $C^2$ such that

$$(8) \quad s'(x+\omega) - s'(x) = -\frac{1}{2}\int_0^{\omega} f(v, x+\omega-v)\,dv = F(x) .$$

(Here we suppose that $f(t,x)$ is of class $C^0$ in $t$, $x \in [0,\pi]$, of class $C^1$ in $x$, $t \in \langle 0, +\infty)$, $f(t,0) = f(t,\pi) = 0$.) Let, in the sequel, $\alpha$ be an irrational number. By comparing the Fourier coefficients of $s'(x)$ and $F(x)$ we obtain from (8) the relations

$$a_k = \frac{c_k}{e^{ik\omega}-1} ,$$

where

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} F(\xi) e^{-ik\xi} d\xi, \quad a_k = \frac{1}{2\pi}\int_0^{2\pi} s'(\xi) e^{-ik\xi} d\xi .$$

Because of

$$|e^{ik\omega} - 1| = 2\left|\sin \pi \left\langle \frac{k\omega}{2\pi} \right\rangle\right| \geq 4 \left\langle \frac{k\omega}{2\pi} \right\rangle ,$$

where $\langle t \rangle$ means the distance of $t$ from the nearest integer, it is easy to see that the sufficient condition for the existence of classical solution of the problem (1) is the convergence of the series

$$\sum_{k=1}^{\infty} \frac{|c_k|}{\langle k\alpha \rangle}$$

Now, using continued fractions (see [1]) one can prove that the series

$$\sum_{k=1}^{\infty} \frac{1}{k^{\rho}\langle k\alpha \rangle} \tag{9}$$

converges for $\rho > \gamma$ and diverges for $\rho < \gamma$ [2]. (If $\rho = \gamma$ the series (9) can either converge or diverge depending on the specific value of $\alpha$.) Applying this assertion one can prove easily the following

<u>Theorem 2.</u> Let the notation introduced above be observed. Then the problem (1) has a classical solution if $c_k = O(k^{-\rho})$, $\rho > \gamma$. Conversely, for $1 \le \rho < \gamma$, $c_k = k^{-\rho}$ the classical solution of (1) does not exist.

<u>Remark</u>. 1) For general $\rho$, the conditions for the solvability of (1) in the terms of $f$ it seems to be difficult to formulate.

2) If $f(t,x)$ is of class $C^{n}$ in $x$, then $c_k = O(k^{-n})$ and thus the problem (1) has the classical solution provided $\gamma < n$.

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2) A very short proof is published in [2], p. 770.

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