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Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 2, 361--375

Persistent URL: `http://dml.cz/dmlcz/105561`

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15,2 (1974)

GENERALIZED SYMMETRIC SPACES x)

(Preliminary communication)

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Abstract: In this note we give some new results concerning generalized symmetric Riemannian spaces (i.e., Riemannian manifolds which admit a regular family of symmetries in the sense of A.J. Ledger). We also present a complete classification of all simply connected irreducible generalized symmetric spaces of dimension 3, 4 and 5 that are not symmetric in the sense of E. Cartan.

Key words: Homogeneous manifolds, Riemannian manifolds, symmetric spaces.

AMS: 53.66, 53.73 Ref. Ž. 3.933.312

Let (M, g) be a smooth Riemannian manifold. An s-structure on (M, g) is a family $\{s_x : x \in M\}$ of isometries of (M, g) (called symmetries) such that each s_x has the point x as an isolated fixed point. The corresponding tensor field S of type $(1,1)$ defined by $S_x = (s_{x*})_x$ for each $x \in M$ is called the symmetry tensor field of $\{s_x\}$. Follow-

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x) A report at the "Tagung über Geometrie", Oberwolfach, September 1973.

With respect to the special character of the paper, Editorial Board agreed with the unusual size of this preliminary communication.

ing A.J. Ledger ([1],[2]) an s-structure $\{s_x\}$ on (M, g) is called <u>regular</u> if for every pair of points x, $y \in M$

$$s_x \circ s_y = s_z \circ s_x \ , \quad \text{where } z = s_x(y) \ .$$

An equivalent condition is that the corresponding tensor field S is invariant by each s_x, i.e., for all $x \in M$ and all vector fields X on M

$$s_{x_*}(SX) = S(s_{x_*}X) \ .$$

From a result by F. Brickel ([3], Theorem 1) we can obtain:

<u>Theorem 1</u>. For a regular s-structure $\{s_x\}$ on (M, g) the symmetry tensor field S is always smooth.

An s-structure $\{s_x\}$ is called <u>of order</u> k ($k \geq 2$) if, for all $x \in M$, $(s_x)^k$ = identity, and k is the least integer of this property.

Using an unpublished result by A.W. Deicke we can prove

<u>Theorem 2</u>. If the Riemannian manifold (M, g) admits a regular s-structure then it also admits a regular s-structure of finite order.

On account of Theorem 2 we introduce

<u>Definition 1</u>. A <u>generalized symmetric space</u> (g.s. space) is a Riemannian manifold (M, g) admitting a regular s-structure. <u>Order</u> of a g.s. space (M, g) is the least integer k such that M admits a regular s-structure of order k.

Let us remark that the g.s. spaces of order 2 are nothing but the Cartan symmetric spaces, and the g.s. spaces of dimension 2 are homogeneous spaces of constant curvature.

Let (M, g) be a g.s. space and $\{s_x\}$ a fixed regular s-structure on (M, g). Then the triplet $(M, g, \{s_x\})$ will be called a (Riemannian) s -manifold. Let now ∇ denote the Riemannian connection of (M, g) and S the symmetry tensor field of $\{s_x\}$. Following A.J. Ledger [1], we introduce a new linear connection $\tilde{\nabla}$ by the formula

$$\tilde{\nabla}_X Y = \nabla_X Y - D(Y, X), \quad \text{where}$$
$$D(Y, X) = (\nabla S)(S^{-1}Y, (I-S)^{-1}X) = (\nabla_{(I-S)^{-1}X} S)(S^{-1}X).$$

The basic properties of the connection $\tilde{\nabla}$ are the following:

1) All symmetries s_x, $x \in M$, are affine transformations of the affine manifold $(M, \tilde{\nabla})$.

2) The affine manifold $(M, \tilde{\nabla})$ is complete.

3) $(M, \tilde{\nabla})$ has parallel curvature and parallel torsion, i.e., $\tilde{\nabla}\tilde{R} = 0$, $\tilde{\nabla}\tilde{T} = 0$.

4) $\tilde{\nabla} S = 0$, $\tilde{\nabla}(\nabla S) = 0$, $\tilde{\nabla} g = 0$.

The next definition brings together all the algebraic compatibility conditions among the tensor fields $g, S, \tilde{R}$ and $\tilde{T}$:

Definition 2. An algebraic s -manifold is a collection $(V, g_0, S_0, \tilde{R}_0, \tilde{T}_0)$, where V is (real) vector space, g_0 is a positive inner product on V, $S_0, \tilde{R}_0, \tilde{T}_0$ are tensors of types $(1,1)$, $(1,3)$, $(1,2)$ respectively, and the following conditions are satisfied:

(i) Both S_0, $I - S_0$ are non-singular transformations of V

(ii) For any $X, Y \in V$ the endomorphism $\widetilde{R}_0(X,Y)$ acting as derivation on the tensor algebra $\mathcal{T}(V)$ satisfies

$$\widetilde{R}_0(X,Y)\widetilde{R}_0 = R_0(X,Y)\widetilde{T}_0 = \widetilde{R}_0(X,Y)g_0 = \widetilde{R}_0(X,Y)S_0 = 0$$

(iii) The tensors $\widetilde{R}_0$, $\widetilde{T}_0$, g_0 are invariant by S_0

(iv) $\widetilde{R}_0(X,Y) = -\widetilde{R}_0(Y,X)$, $\widetilde{T}_0(X,Y) = -\widetilde{T}_0(Y,X)$

(v) The first Bianchi identity $\sigma(\widetilde{R}_0(X,Y)Z - \widetilde{T}_0(\widetilde{T}_0(X,Y),Z)) = 0$ holds

(vi) The second Bianchi identity $\sigma(\widetilde{R}_0(\widetilde{T}_0(X,Y),Z)) = 0$ holds.

We shall make use of the following theorem by A.J. Ledger ([1]):

<u>Theorem A</u>. Let $(M, g, \{s_x\})$ be an s-manifold. Then the group of all isometries of (M, g) keeping the tensor field S invariant is a transitive Lie group. Hence (M, g) is a homogeneous Riemannian manifold G/H and it is a complete Riemannian space.

On account of this theorem we can make

<u>Definition 3</u>. Two s-manifolds $(M, g, \{s_x\})$, $(M', g', \{s'_x\})$ are called <u>locally isomorphic</u> if for any two points $p \in M$, $p' \in M'$ there is an isometry ϕ of a neighbourhood $U \ni p$ onto a neighbourhood $U' \ni p'$ such that $\phi_*(S|_U) = S'|_{U'}$.

<u>Definition 4</u>. Two algebraic s-manifolds $(V_i, g_i, S_i, \widetilde{R}_i, \widetilde{T}_i)$ $i = 1, 2$ will be called <u>isomorphic</u> if there is a linear isomorphism $f : V_1 \longrightarrow V_2$ of vector spaces such that $f(g_1) = g_2$,

$f(S_1) = S_2$, $f(\widetilde{R}_1) = \widetilde{R}_2$, $f(\widetilde{T}_1) = \widetilde{T}_2$.

Theorem 3. Let $(M, g, \{s_x\})$ be an s-manifold. Then for each point $p \in M$ the collection $(M_p, g_p, S_p, \widetilde{R}_p, \widetilde{T}_p)$ is an algebraic s-manifold and for any two points $p, q \in M$ the corresponding algebraic s-manifolds are isomorphic.

We shall call the isomorphism class of all $(M_p, g_p, S_p, \widetilde{R}_p, \widetilde{T}_p)$, $p \in M$, the algebraic type of the s-manifold $(M, g, \{s_x\})$.

Theorem 4 (Equivalence theorem). Two s-manifolds are locally isomorphic if and only if they have the same algebraic type.

Notice that two locally isomorphic simply connected s-manifolds are globally isomorphic.

Using a construction by K. Nomizu ([4]), we can prove

Theorem 5 (Existence theorem). Any algebraic s-manifold is the algebraic type of a (unique) simply connected s-manifold.

We have also the following result by A.J. Ledger, which corresponds to Theorem 6.2 of [1].

Theorem B. For any s-manifold there is a simply connected covering s-manifold such that the covering map is a local isomorphism in the neighbourhood of each point.

The following result is useful in all kinds of classification problems:

Theorem 6. Let (M, g) be a simply connected g.s. space and let $M = M_0 \times M_1 \times \dots \times M_r$ be the de Rham decomposition of (M, g) (i.e., M_0 is the Euclidean part and $M_1, \dots, M_r$ are irreducible components). Then all Riemannian spaces $M_0, M_1, \dots, M_r$ are g.s. spaces. Moreover, any regular s-structure of order k on (M, g) determines a regular s-structure of order k_i on each M_i, where $k_i | k$ for $i = 0, 1, \dots, r$.

A modest classification problem.

According to Theorem 5, if we succeed in classifying all algebraic s-manifolds of a given dimension, then we can classify all simply connected s-manifolds and thus all simply connected generalized symmetric spaces of this dimension.

In the rest of this note we present a complete classification of all simply connected and irreducible g.s. spaces of dimensions 3, 4, 5 and of orders greater than 2 (we shall call these spaces briefly "exceptional" ones). It means, we leave out all symmetric spaces of E. Cartan which are well-known. In each case we shall give a representation in the form of a homogeneous Riemannian space (cf. Theorem A). As a rule, we shall describe first the underlying homogeneous manifold and then we give the family of all admissible invariant metrics in a different, more explicit form.

The details of the method and the complete proofs will appear as a special issue in the edition "Rozpravy ČSAV", Czechoslovak Academy of Sciences, Prague.

Dimension $n = 3$.

All exceptional spaces are of order 4 and of the following type:

As a homogeneous space, M is the matrix group $\begin{Vmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{Vmatrix}$.

Also, M is the space $\mathbb{R}^3(x, y, z)$ with a Riemannian metric $g = e^{2z}dx^2 + e^{-2z}dy^2 + \lambda^2 dz^2$, where $\lambda > 0$ is a constant.

The typical symmetry at the point $(0, 0, 0)$ is the transformation $x' = -y,\ y' = x,\ z' = -z$.

Dimension $n = 4$.

All exceptional spaces are of order 3 and of the following type:

M is the homogeneous space $\begin{Vmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{Vmatrix} \Big/ \begin{Vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{Vmatrix}$

where $\det \begin{Vmatrix} a & b \\ c & d \end{Vmatrix} = 1$.

Also, M is the space $\mathbb{R}^4(x, y, u, v)$ with a Riemann metric

$$g = (-x + \sqrt{x^2 + y^2 + 1})\,du^2 + (x + \sqrt{x^2 + y^2 + 1})\,dv^2 -$$

$$- 2y\,du\,dv + \lambda^2 \cdot \frac{(1+y^2)dx^2 + (1+x^2)dy^2 - 2xy\,dx\,dy}{1 + x^2 + y^2} \quad (\lambda > 0)$$

Each transformation $u' = \cos t \cdot u - \sin t \cdot v,\ x' = \cos 2t \cdot x - \sin 2t \cdot y$

$$v' = \sin t . u + \cos t . v, \quad y' = \sin 2t . x + \cos 2t . y$$

for $t \neq k\pi$ is a symmetry at the point $(0,0,0,0)$ which extends to a regular s-structure on M.

Dimension $n = 5$.

All exceptional spaces are of order 4 or 6, and of the following 12 types:

Type 1.

As a homogeneous space, M is the matrix group $\begin{Vmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ u & v & 1 & z \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

Also, M is the space $R^5(x,y,z,u,v)$ with a Riemann metric $g = dx^2 + dy^2 + dv^2 + \lambda^2 (x\,du + y\,dv - dz)^2 \quad (\lambda > 0)$. The typical symmetry at the point $(0, \dots, 0)$ is the transformation $x' = y$, $y' = -x$, $z' = -z$, $u' = -v$, $v' = u$.

Type 2.

M is the matrix group depending of two real parameters $\lambda_1 > 0$, $\lambda_2 \geq 0$: $\begin{Vmatrix} e^{\lambda_1 t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda_1 t} & 0 & 0 & y \\ 0 & 0 & e^{\lambda_2 t} & 0 & z \\ 0 & 0 & 0 & e^{-\lambda_2 t} & w \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$

Also, M is the space $R^5(x,y,z,w,t)$ with a Riemann metric

$$g = e^{-2\lambda_1 t} dx^2 + e^{2\lambda_1 t} dy^2 + e^{-2\lambda_2 t} dz^2 + e^{2\lambda_2 t} dw^2 + dt^2 +$$
$$+ 2\alpha \left[e^{-(\lambda_1+\lambda_2)t} dx\,dz + e^{(\lambda_1+\lambda_2)t} dy\,dw \right] +$$

$$+ 2\beta \left[e^{(\lambda_1-\lambda_2)t} dy\,dz - e^{(\lambda_2-\lambda_1)t} dx\,dw\right] .$$

Here either $\lambda_1 > \lambda_2 > 0$, $\alpha^2 + \beta^2 < 1$, or $\lambda_1 = \lambda_2 > 0$, $\alpha = 0,\ 0 \leq \beta < 1$, or $\lambda_1 > 0$, $\lambda_2 = 0$, $\alpha = 0$, $0 < \beta < 1$. The typical symmetry at the point $(0,\dots,0)$ is the transformation $x' = -y,\ y' = x,\ z' = -w,\ w' = z,\ t' = -t$.

<u>Type 3</u>. M is the homogeneous space $SO(3,C)/SO(2)$, where $SO(3,C)$ denotes the special complex orthogonal group and $SO(2)$ denotes the subgroup $\left\| \begin{array}{c|c} SO(2) & 0 \\ \hline 0 & 1 \end{array} \right\|$ of $SO(3,C)$.

The Riemann metric g in M is induced by the following real invariant positive semi-definite form on the group $GL(3,C)$ of all regular complex matrices $\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|$:

$$\tilde{g} = \lambda^2(\omega_1\bar{\omega}_1 + \omega_2\bar{\omega}_2) + \gamma(\omega_1^2 + \bar{\omega}_1^2 + \omega_2^2 + \bar{\omega}_2^2) + \mu^2\left(\frac{\omega_3 - \bar{\omega}_3}{i}\right)^2$$

where $\omega_1 = a_2\,da_3 + b_2\,db_3 + c_2\,dc_3$, $\omega_2 = a_3\,da_1 + b_3\,db_1 + c_3\,dc_1$, $\omega_3 = a_1\,da_2 + b_1\,db_2 + c_1\,dc_2$, and λ, γ, μ are real parameters satisfying $\lambda > 0,\ \mu > 0,\ |2\gamma| < \lambda^2$.

The typical symmetry at the origin of M is induced by the following transformation of $GL(3,C)$:

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} \longrightarrow \begin{Vmatrix} \bar{b}_2 & -\bar{b}_1 & \bar{b}_3 \\ -\bar{a}_2 & \bar{a}_1 & -\bar{a}_3 \\ \bar{c}_2 & -\bar{c}_1 & \bar{c}_3 \end{Vmatrix} .$$

Type 4.

M is the complex matrix group depending of a complex parameter λ : $\begin{Vmatrix} e^{\lambda t} & 0 & x \\ 0 & e^{-\lambda t} & w \\ 0 & 0 & 1 \end{Vmatrix}$ Here x, w denote complex variables and t a real variable.

Also, M is the space $C^2(x,w) \times R^1(t)$ with a (real) Riemann metric

$$g = e^{-(\lambda+\bar\lambda)t} dx d\bar{x} + e^{(\lambda+\bar\lambda)t} dw d\bar{w} + (dt)^2 + 2\mu \left[e^{(\bar\lambda-\lambda)t} dx d\bar{w} + \right.$$

$$\left. + e^{(\lambda-\bar\lambda)t} d\bar{x} dw \right] + \alpha e^{-2\lambda t}(dx)^2 + \bar\alpha e^{-2\bar\lambda t}(d\bar{x})^2 - \alpha e^{2\lambda t}(dw)^2 - \bar\alpha e^{2\bar\lambda t}(d\bar{w})^2 .$$

Here λ, α are complex parameters, μ a real parameter, $\alpha\bar\alpha + \mu^2 < 1/4$. In case that $\lambda + \bar\lambda = 0$, $\mu = 0$ we have $\alpha = 0$.

The typical symmetry at the point $(0,0;0)$ is the transformation $x' = iw$, $w' = ix$, $t' = -t$.

Types 5a, 5b.

M is the homogeneous space $SO(3)\times SO(3)/SO(2)$, or $SO(2,1)\times SO(2,1)/SO(2)$, where $SO(2)$ denotes the subgroup $\begin{Vmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{Vmatrix}$

The Riemann metric g is induced by the following real invariant positive semi-definite form on the group $GL(3,R)\times \times GL(3,R)$ of all non-singular product matrices

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} \times \begin{Vmatrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{Vmatrix} :$$

$$\tilde{g} = \alpha^2\left[(\omega_1+\tilde{\omega}_2)^2+(\tilde{\omega}_1+\omega_2)^2\right] + \beta^2\left[(\omega_1-\tilde{\omega}_2)^2+(\tilde{\omega}_1-\omega_2)^2\right] + \\ + \gamma^2(\omega_3+\tilde{\omega}_3)^2 ,$$

where $\omega_1 = a_2 da_3 + b_2 db_3 \pm c_2 dc_3$ and $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ are given
$\omega_2 = a_3 da_1 + b_3 db_1 \pm c_3 dc_1$ by similar expressions in
$\omega_3 = a_1 da_2 + b_1 db_2 \pm c_1 dc_2$ $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i, d\tilde{a}_i, d\tilde{b}_i, d\tilde{c}_i$.

Here α, β, γ are positive real parameters, $\alpha \geq \beta$, and the $(+)$ and $(-)$ signs in $\omega_1, \omega_2, \omega_3$ correspond to the elliptic case 5a and to the hyperbolic case 5b respectively.

The typical symmetry at the origin of M is induced by the following transformation of $GL(3,R)\times GL(3,R)$:

$$\begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix} \times \begin{Vmatrix} \tilde{a}_1 & \tilde{a}_2 & \tilde{a}_3 \\ \tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{Vmatrix} \longrightarrow \begin{Vmatrix} \tilde{a}_1 & -\tilde{a}_2 & -\tilde{a}_3 \\ -\tilde{b}_1 & \tilde{b}_2 & \tilde{b}_3 \\ -\tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{Vmatrix} \times \begin{Vmatrix} a_1 & -a_2 & a_3 \\ -b_1 & b_2 & -b_3 \\ c_1 & -c_2 & c_3 \end{Vmatrix}$$

Types 6a, 6b.

M is the homogeneous space $SU(3)/SU(2)$, or $SU(2,1)/SU(2)$. Also, M is the submanifold of $C^3(x^1,x^2,x^3)$ given by the relation $x^1\bar{x}^1 + x^2\bar{x}^2 \pm x^3\bar{x}^3 = \pm 1$. The Riemann metric on M is induced by the following hermitian metric on C^3:

$$\tilde{g} = \lambda(dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2 \pm dz^3 d\bar{z}^3) +$$
$$+ \mu(z^1 d\bar{z}^1 + z^2 d\bar{z}^2 \pm z^3 d\bar{z}^3)(\bar{z}^1 dz^1 + \bar{z}^2 dz^2 \pm \bar{z}^3 dz^3)$$

where λ, μ are real parameters such that $\lambda > 0$, $\mu \neq 0$ and $\mu \pm \lambda > 0$. The $(+)$ and $(-)$ signs correspond to the elliptic case 6a and to the hyperbolic 6b respectively.

The typical symmetry at the point $(0, 0, 1)$ of M is induced by the following transformation of C^3:

$$z^{1\prime} = \bar{z}^2, \quad z^{2\prime} = -\bar{z}^1, \quad z^{3\prime} = \bar{z}^3 .$$

<u>Remark</u>. The case 6a was communicated to me orally by A.W. Deicke.

<u>Type 7</u>.

M is the real matrix group (t, x, y, u, v) are real variables and λ is a real parameter).

$$\begin{Vmatrix} e^{\lambda t} & 0 & 0 & 0 & x \\ 0 & e^{-\lambda t} & 0 & 0 & y \\ te^{\lambda t} & 0 & e^{\lambda t} & 0 & u \\ 0 & -te^{-\lambda t} & 0 & e^{-\lambda t} & v \\ 0 & 0 & 0 & 0 & 1 \end{Vmatrix}$$

Also, M is the space $R^5(x, y, u, v, t)$ with a Riemann metric $g = (dt)^2 + e^{-2\lambda t}(t\,dx - du)^2 + e^{2\lambda t}(t\,dy + dv)^2 +$
$$+ \mu(e^{-2\lambda t}dx^2 + e^{2\lambda t}dy^2) + 2\gamma(dy\,du - dx\,dv),$$

where λ, μ, γ are real parameters, $\lambda \geq 0$, $\mu > 0$, $\gamma^2 < \mu$.

The typical symmetry at the point $(0, \ldots, 0)$ is the transformation $x' = -y$, $y' = x$, $u' = -v$, $v' = u$, $t' = -t$.

<u>Types 8a, 8b.</u>

M is the homogeneous space $I^e(R^3)/SO(2)$ or

$I^h(R^3)/SO(2)$, where $I^e(R^3)$, or $I^h(R^3)$, denotes the group of all positive affine transformations of the space $R^3(x,y,z)$ that preserve the differential form $dx^2 + dy^2 + dz^2$, or $dx^2 + dy^2 - dz^2$, respectively. ($I^e(R^3)$ is the semidirect product of $SO(3)$ and $t(3)$ and $I^h(R^3)$ is the semidirect product of $SO(2,1)$ and $t(3)$, where $t(3)$ denotes the translation group of R^3 .)

Also, M is the submanifold of $R^6(x,y,z;\alpha,\beta,\gamma)$ given by the relation $\alpha^2+\beta^2 \pm \gamma^2 = \pm 1$. The Riemann metric on M is induced by the following non-singular invariant quadratic form on R^6 :

$$\tilde{g} = dx^2 + dy^2 \pm dz^2 + \lambda^2(d\alpha^2 + d\beta^2 \pm d\gamma^2) + \\ + \left[\mu^2 \pm (-1)\right](\alpha\, dx + \beta\, dy \pm \gamma\, dz)^2$$

where $\lambda > 0$, $\mu > 0$ are real parameters. The $(+)$ and $(-)$ signs correspond to the elliptic case 8a and to the hyperbolic case 8b respectively. In the elliptic case $\mu \neq 1$.

The typical symmetry at the point $(0,0,0;0,1)$ of M is induced by the following transformation of R^6 :

$$x' = -y,\ y' = x,\ z' = -z,\ \alpha' = \beta,\ \beta' = -\alpha,\ \gamma' = \gamma \ .$$

All preceding exceptional spaces (types 1 - 8b) are of order 4 .

<u>Type 9</u>. (Spaces of order 6 .)

M is the matrix group $\begin{Vmatrix} e^{-(u+v)} & 0 & 0 & x \\ 0 & e^{u} & 0 & y \\ 0 & 0 & e^{v} & z \\ 0 & 0 & 0 & 1 \end{Vmatrix}$

Also, M is the space $R^5(x, y, z, u, v)$ with a Riemann metric

$$g = a^2(du^2 + dv^2 + du\,dv) + (b^2+1)(e^{2(u+v)}dx^2 + e^{-2u}dy^2 + e^{-2v}dz^2) + (b^2-2)(e^{v}dx\,dy + e^{u}dx\,dz - e^{-(u+v)}dy\,dz),$$

where $a > 0$, $b > 0$.

The typical symmetry at the point $(0,\dots,0)$ is the transformation $x' = y$, $y' = -z$, $z' = x$; $u' = v$, $v' = -(u+v)$.

To conclude, let us remark that two Riemann spaces of different types are always non-isometric and within each type, the corresponding parameters are invariants of the Riemann metric.

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(Oblatum 23.2.1974)