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Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 2, 341--344

Persistent URL: http://dml.cz/dmlcz/105557

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15,2 (1972)

COMPLETELY ADDITIVE DISJOINT SYSTEM OF BAIRE SETS IS OF BOUNDED CLASS

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Abstract: The theorem in the title is proved; this result corrects the proof of Lemma 2 in [F], and thus makes all deep results in [F] verified.

Key words: Completely additive systems, Baire classification of sets.

AMS: Primary 28A05 Ref. Ž. 7.518.11

Secondary 26A21

1. Frolík [F] claimed to prove several deep results on completely metrizable spaces concerning Baire measurable maps and maps of bounded class. His proofs are based on Hansel's lemma (see [F]) and on its converse (see Lemma 2 [F , p.140]) which gives a characterization of disjoint Baire completely additive systems. But the proof of Lemma 2 in [F] does not seem to be correct (Why should the sets X'_n be closed in X' ?). To give a correct proof of this lemma it would be sufficient to answer positively the following question: If $\{X_a \mid a \in A\}$ is a disjoint family in an absolute Souslin space such that the union of each subfamily of $\{X_a\}$ is a Baire set, is it true that the family $\{X_a\}$ ranges in some Baire class?

In this note we prove that the answer is yes even in more general setting. In spite of what was said before it would follow that the theorems in F hold.

2. Let $\mathcal{X}$ be any set and let $\mathcal{B}_0$ be any family of subsets of $\mathcal{X}$. Let $\mathcal{B} = \{\mathcal{B}_\alpha \mid \alpha < \omega_1\}$ be defined as follows:

$\mathcal{B}_\alpha$ is the collection of all countable unions or intersections of elements of $\bigcup \{\mathcal{B}_\beta \mid \beta < \alpha\}$ according to as α is odd or even.

For $B \in \mathcal{B}$ put class $B = \min \{\alpha \mid B \in \mathcal{B}_\alpha\}$.

As an example we can have $\mathcal{X}$ a topological (or uniform) space and $\mathcal{B}_0$ the family of zero sets.

Our result can be expressed now as follows.

<u>Theorem</u>. Let $\{X_a \mid a \in A\}$ be a disjoint family of subsets of $\mathcal{X}$. If the union of each subfamily of $\{X_a\}$ belongs to $\mathcal{B}$ then the family $\{X_a\}$ ranges in some $\mathcal{B}_\alpha$.

3. A limit ordinal number ε is called regular if any co-final subset of T_ε is of the type ε.

<u>Lemma</u>. Let ε be a regular ordinal number. Let A be any set and φ a map of $\exp A$ into T_ε. Suppose that there is a map $\eta : T_\varepsilon \times T_\varepsilon \longrightarrow T_\varepsilon$ such that:

$$\varphi(A_1 \cap A_2) \leqq \eta(\varphi(A_1), \varphi(A_2)) .$$

Then there is an $\alpha < \varepsilon$ such that $\varphi(a) \leqq \alpha$ for any $a \in A$.

<u>Proof</u>. Suppose that the family $\{\varphi(a) \mid a \in A\}$ is

not bounded by any $\alpha < \varepsilon$. By induction we can, for any $\alpha < \varepsilon$, choose an $a_\alpha \in A$ such that

$$\varphi(a_\alpha) \geqq \alpha , \quad \varphi(a_\alpha) > \sup\{\varphi(a_\beta) \mid \beta < \alpha\} .$$

(If a_α have been defined for $\alpha < \alpha_0 < \omega$, then $\sup\{\varphi(a_\alpha) \mid \alpha < \alpha_0\} < \varepsilon$ since ε is regular.) Using the correspondence between α and a_α we see that we only have to find a contradiction in case $A = T_\varepsilon$ and $\varphi(\alpha) \geqq \alpha$ for any $\alpha < \varepsilon$.

Let K_0 be the set of all limit ordinals $< \varepsilon$ and let K_α be the set of all limit members of $\bigcap_{\beta<\alpha} K_\beta$ in itself $(\alpha < \varepsilon)$. For any $\alpha < \varepsilon$ the set $\bigcap_{\beta<\alpha} K_\beta$ is a closed subset of T_ε. By induction we easily show that this set is also co-final. (If this holds for any $\alpha < \alpha_0 < \varepsilon$ then, for given $\gamma < \varepsilon$, put $\gamma_\beta = \min(K_\beta \cap (T_\varepsilon \setminus T_\gamma))$. Since the set $\{\gamma_\beta \mid \beta < \alpha_0\}$ is not co-final, we have

$$\sup\{\gamma_\beta \mid \beta < \alpha_0\} \in \bigcap_{\beta<\alpha} K_\beta \ .)$$

Now choose $\alpha_\beta \in \bigcap_{\gamma<\beta} K_\gamma \setminus K_\beta$ such that

$$\varphi(\alpha_\beta) > \eta(\beta, \varphi(\bigcap_{\gamma<\beta} K_\gamma \setminus K_\beta)) .$$

(We put $\bigcap_{\gamma<0} K_\gamma = T_\varepsilon$.)
(The existence of α_β follows from the co-finality of the sets $\bigcap_{\gamma<\beta} K_\gamma \setminus K_\beta$.)

Since $\varphi(\alpha_\beta) = \varphi(\{\alpha_\gamma \mid \gamma < \varepsilon\} \cap (\bigcap_{\gamma<\beta} K_\gamma \setminus K_\beta)) \leqq$

$$\leqq \eta(\varphi\{\alpha_\gamma \mid \gamma < \varepsilon\}, \ \varphi(\bigcap_{\gamma<\beta} K_\gamma \setminus K_\beta))$$

we obtain the contradiction putting $\beta = \varphi(\{\alpha_\gamma \mid \gamma < \varepsilon\})$.

Remark. The preceding lemma holds for arbitrary ordinal numbers if we suppose that η is monotone (i.e. $\eta(\alpha,\beta) \le \eta(\alpha',\beta')$ if $\alpha \le \alpha'$ and $\beta \le \beta'$). For non-limit ordinals the proof is obvious. To prove it for a limit ordinal ε one finds a co-final subset T of T_ε of the smallest possible type and defines $\bar{\varphi}(B) = \min\{\gamma \in T \mid \varphi(B) \le \gamma\}$ and $\bar{\eta}(\alpha,\beta) = \inf\{\gamma \in T \mid \eta(\alpha,\beta) \le \gamma\}$ for $\alpha,\beta \in T$. Using Theorem 2 (where T_ε is replaced by T) one obtains the proof.

Proof of the Theorem. Put, for $B \subset A$, $\varphi(B) = \mathrm{class}(\bigcup_{a \in B} X_a)$. Using the lemma for $\varepsilon = \omega_1$ (put $\eta(\alpha,\beta) = \max(\alpha,\beta) + 1$) we obtain that the family $\{\varphi(a) \mid a \in A\}$ is bounded, i.e. $\{X_a \mid a \in A\}$ ranges in some $\mathcal{B}_\alpha$.

R e f e r e n c e

[F] FROLÍK Z.: Baire sets and uniformities on complete metric spaces, Comment.Math.Univ.Carolinae 13 (1972), 137-147.

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(Oblatum 2.5.1974)