Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 15 (1974), No. 2, 259--271

Persistent URL: `http://dml.cz/dmlcz/105550`

COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

15,2 (1974)

FURTHER REMARK ON A THEOREM BY E.M.LANDESMAN AND A.C. LAZER

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Abstract: It is considered the existence of the solution of the boundary value problem $\Delta u + \lambda u + g(u) = h$ on Ω , $u|_{\partial\Omega} = 0$ in the case $\lim_{s \to +\infty} g(s) = \lim_{s \to -\infty} g(s) = 0$ and under some other additional conditions.

Key words: Equations involving nonlinear operators, boundary value problems for ordinary and partial differential equations, weak and classical solutions, positive eigenfunctions.

AMS, Primary: 47H15 Ref. Ž. 7.798

Secondary 34B15, 35J60

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1. Introduction. Let Ω be a bounded domain in R_N $(N \geq 1)$ and suppose

$$Lu = \sum_{|\alpha|,|\beta| \leq 1} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta} u)$$

is a uniformly elliptic, formally selfadjoint linear differential expression defined on Ω , with real-valaed coefficients $a_{\alpha\beta} = a_{\beta\alpha} \in L_{\infty}(\Omega)$. Let further $g : R_1 \to R_1$ be a bounded continuous function. We are concerned with the solvability of the Dirichlet problem

$$(1) \qquad \begin{cases} (Lu)(x) = g(u(x)) - h(x), & x \in \Omega \\ u(x) = 0 & , \quad x \in \partial\Omega \end{cases}$$

for a given real-valued function h .

In the series of papers (see e.g. [3],[6],[10],...) they are given sufficient conditions for the function g to be the boundary value problem (1) weakly solvable for each "right hand side" $h \in L_2(\Omega)$ provided the linear boundary value problem

$$(2) \qquad \begin{cases} (Lu)(x) = 0 , & x \in \Omega \\ u(x) = 0 , & x \in \partial\Omega \end{cases}$$

has only trivial weak solution.

The question about the weak solvability of (1) in the case of the existence of nontrivial solution of the homogeneous linearized problem (2) is more complicated. In [9] it is proved that if 0 is a single eigenvalue of (2) and the null-space of the operator L is spanned by the vector $w\ (\neq 0)$ then under the assumptions

$$(3) \qquad \lim_{s \to +\infty} g(s) = g(+\infty) ,$$

$$(4) \qquad \lim_{s \to -\infty} g(s) = g(-\infty) ,$$

$$(5) \qquad g(-\infty) < g(s) < g(+\infty) \qquad \text{for each } s \in R_1 .$$

the necessary and sufficient condition for the existence of a weak solution of the problem (1) with $h \in L_2(\Omega)$ is the validity of the inequalities

$$(6) \qquad g(-\infty)\int_{\Omega_+}|w(x)|\,dx - g(+\infty)\int_{\Omega_-}|w(x)|\,dx < \int_{\Omega} h(x)w(x)\,dx <$$
$$< g(+\infty)\int_{\Omega_+}|w(x)|\,dx - g(-\infty)\int_{\Omega_-}|w(x)|\,dx ,$$

where $\Omega_{\pm} = \{x \in \Omega ; w(x) \gtrless 0\}$.

Short and elementary proof is given in [7]. This result was generalized in [15] to the case of multiple eigenvalue of the operator L . The assumption (5) is replaced by

$$(7) \quad g(-\infty) \leq g(s) \leq g(+\infty),\ g(0) \neq g(-\infty),\ g(0) \neq g(+\infty)$$

in the paper [5]. The abstract setting of the method from the papers [9],[15] is given in [1],[4],[5],[11] - [14]. Moreover, in these papers the higher order equations and also the more general nonlinear perturbations are considered.

In all previously referred papers about the solvability of (1) it is substantial that the limits $g(+\infty)$, $g(-\infty)$ are different from zero. The purpose of this note is to prove that under some assumptions in the case $g(+\infty) = g(-\infty) = 0$ there exists a solution of (1) for each h which is orthogonal in $L_2(\Omega)$ to the eigenfunction w (see Sections 3 and 4). The proof is based on the abstract theorem which is mentioned in Section 2 (this section serves also as a preliminary communication for a paper [4]).

2. Abstract result

The proof of the result from this section immediately follows from [4, Theorem 2.3.10].

Assumptions: I. Let H be a real Hilbert space with the inner product $(\cdot , \cdot)$ and the norm $\|\cdot\|_H$. Let

$(X, \|\cdot\|_X)$, $(Z, \|\cdot\|_Z)$ be Banach spaces. Suppose $X \subset \subset Z \subset H$.

II. Let $L : \mathrm{Dom}\,[L] \subset X \to Z$ be a closed linear operator with the domain $\mathrm{Dom}\,[L]$ dense in the space X and closed image $\mathrm{Im}\,[L]$ in the space Z and with a finitedimensional null-space $\mathrm{Ker}\,[L]$. Moreover, suppose $Z = \mathrm{Im}\,[L] \oplus \mathrm{Ker}\,[L]$.

(Particularly, it means that L is a Fredholm operator the index of which is equal to zero.)

III. Let $N : X \to Z$, $\mathrm{Dom}\,[N] = X$, be a completely continuous nonlinear mapping such that

$$\sup_{x \in X} \| N(x) \|_Z < +\infty \quad .$$

IV. Suppose that for arbitrary $\kappa > 0$ there exists an interval $\langle \varepsilon_1, \varepsilon_2 \rangle$, $0 < \varepsilon_1 < \varepsilon_2$, such that

$$\inf \{ (N(tw + v), w) \; ; \; v \in X , \; \|v\|_X \le \kappa ,$$

$$w \in \mathrm{Ker}\,[L] , \; \|w\|_H = 1 , \; t \in \langle \varepsilon_1, \varepsilon_2 \rangle \} > 0 \quad .$$

Theorem 1. Let $h \in Z$. Then the equation

$$L(u) = N(u) - h$$

is solvable in $\mathrm{Dom}\,[L]$ provided $(h, w) = 0$ for each $w \in \mathrm{Ker}\,[L]$.

Remark 1. Under the same assumptions (and with the same proof) we can strengthen the assertion of Theorem 1 in the following way:

For each h_0 orthogonal to $\mathrm{Ker}\,[L]$ in H there exists an open neighborhood $U(h_0) \subset Z$ of the point h_0 such that the equation $L(u) = N(u) - h$ is solvable in

$\mathrm{Dom}[L]$ for any $h \in U(h_0)$.

3. <u>Main result</u>

Let Ω be a bounded domain in R_N . Let $H = L_2(\Omega)$ with the usual inner product. Suppose that X and Z are the Banach spaces satisfying the condition I. Moreover, let X be continuously imbedded into $L_\infty(\Omega)$.

(A) Let L be the linear operator satisfying II. Assume that the dimension of $\mathrm{Ker}[L]$ is one and let $\mathrm{Ker}[L]$ be spanned by $w_0 \in \mathrm{Ker}[L]$, $\|w_0\|_{L_2(\Omega)} = 1$ and the Lebesgue measure of the set $\{x \in \Omega ; w_0(x) \leq 0\}$ is zero.

(B) Let N be a nonlinear mapping satisfying III. Suppose that there exists a bounded continuous function $g: R_1 \to R_1$ such that for any $u, v \in X$ it is

$$(N(u), v) = \int_\Omega g(u(x))\, v(x)\, dx \quad .$$

Let $M = \sup_{\xi \in R_1} |g(\xi)| > 0$.

For $\xi > 0$ denote

$\Gamma(\xi) = \{x \in \Omega ; w_0(x) < \xi\}$. Suppose that $\mathrm{meas}\,\Gamma(x) > 0$ if $x > 0$.

(C) Let the function g introduced in (B) satisfy the following condition:

There exists $\eta > 0$ such that

$$\lim_{x \to 0+} \frac{G(\frac{1}{x})}{\mathrm{meas}\,\Gamma(x)} > \frac{M}{\mathrm{meas}\,\Omega} \quad ,$$

$$\lim_{x \to 0+} \frac{H(-\frac{1}{x})}{\mathrm{meas}\,\Gamma(x)} < -\frac{M}{\mathrm{meas}\,\Omega} \quad ,$$

where

$$G(\xi) = \min_{t \in \langle \eta, \xi \rangle} g(t) , \quad \xi > \eta ,$$

$$H(\xi) = \max_{t \in \langle \xi, -\eta \rangle} g(t) , \quad \xi < -\eta .$$

Theorem 2. Under the assumptions (A) – (C) the equation $L(u) = N(u) - h$ is solvable in $Dom[L]$ provided $h \in Z$ and $\int_\Omega h(x)\, w_0(x)\, dx = 0$.

Proof. From the assumption (C) it follows that $g(\xi) > 0$ for $\xi > \eta$ and $g(\xi) < 0$ for $\xi < -\eta$. According to the result from Section 2 it is sufficient to prove the following assertion ($*$):

Let $\varkappa > 0$. Then there exist $\omega > 0$ and an interval $\langle \varepsilon_1 , \varepsilon_2 \rangle$, $0 < \varepsilon_1 < \varepsilon_2$, such that

$$\int_\Omega g(t w_0(x) + v(x))\, w_0(x)\, dx > \omega ,$$

$$\int_\Omega g(-t w_0(x) + v(x))\, w_0(x)\, dx < -\omega$$

for each $t \in \langle \varepsilon_1 , \varepsilon_2 \rangle$ and $v \in X$, $\|v\|_{L_\infty(\Omega)} \le \varkappa$.

Denote $a = \|w_0\|_{L_\infty(\Omega)}$. Since

$$\lim_{x \to 0_+} meas\, \Gamma(x) = 0$$

there exists $\delta > 0$ and $\mu > 0$ such that

$$meas\, \Gamma(\Delta) < \frac{1}{2}\, meas\, \Omega$$

and

$$\frac{G(\frac{1}{\Delta})}{meas\, \Gamma(\Delta)} - \frac{M}{meas\, \Omega - meas\, \Gamma(\Delta)} \ge \mu ,$$

$$\frac{H(-\frac{1}{\Delta})}{meas\ \Gamma(\Delta)} + \frac{M}{meas\ \Omega - meas\ \Gamma(\Delta)} \le -\mu ,$$

where

$$\Delta = \frac{\delta}{2(\eta+\varkappa)a + \varkappa\delta}$$

Set

$$\varepsilon_1 = \frac{\eta+\varkappa}{\delta}, \quad \varepsilon_2 = 2\,\frac{\eta+\varkappa}{\delta}, \quad \omega = \frac{\mu}{2}\,\Delta\, meas\ \Omega\ meas\ \Gamma(\Delta) .$$

Thus we have for $t \in \langle \varepsilon_1, \varepsilon_2 \rangle$, $v \in X$ with $\|v\|_{L_\infty(\Omega)} \le \varkappa$ the following relations

$$\int_\Omega g(tw_0(x) + v(x))\, w_0(x)\,dx = \int_{\Omega\setminus\Gamma(\Delta)} \% + \int_{\Gamma(\Delta)} \% \ge$$

$$\ge G(\tfrac{1}{\Delta})\,\Delta\,(meas\ \Omega - meas\ \Gamma(\Delta)) - M\,\Delta\ meas\ \Gamma(\Delta) =$$

$$= \Delta\,(meas\ \Omega - meas\ \Gamma(\Delta))\,meas\ \Gamma(\Delta)\left[\frac{G(\frac{1}{\Delta})}{meas\ \Gamma(\Delta)} - \frac{M}{meas\ \Omega - meas\ \Gamma(\Delta)}\right] \ge$$

$$\ge \frac{\mu}{2}\,\Delta\, meas\ \Omega\ meas\ \Gamma(\Delta) = \omega$$

and

$$\int_\Omega g(-tw_0(x) + v(x))\, w_0(x)\,dx = \int_{\Omega\setminus\Gamma(\Delta)} \% + \int_{\Gamma(\Delta)} \% \le$$

$$\le H(-\tfrac{1}{\Delta})\,\Delta\,(meas\ \Omega - meas\ \Gamma(\Delta)) + M\,\Delta\ meas\ \Gamma(\Delta) \le -\omega .$$

<u>Remark 2</u>. If $meas\ \Gamma(x) = 0$ for $x \in (0, \delta_0)$ then the assertion of Theorem 2 is valid if the assumption (C) is replaced by (C′):

There exists $\eta > 0$ such that $g(\xi) > 0$, $g(-\xi) < 0$ for $\xi > \eta$.

(The proof is the same as the previous one.)

4. Applications

Firstly we note that the investigation of the problem about the existence of simple eigenvalue of the linear mapping such that the corresponding eigenvalue is nonnegative is included e.g. in the book [8], where also the special cases of linear integral and differential operators are studied.

In this section we give the applications of Theorem 2 to the existence of a weak solution of boundary value problems for ordinary differential equations and to the existence of a classical solution of boundary value problems for partial differential equations.

a) Weak solutions of boundary value problems for ordinary differential equations.

Set $X = \overset{\circ}{W}{}_2^1(0,\pi) = Z$ the Sobolev space of all absolutely continuous functions u on the interval $(0,\pi)$ with $u(0)=u(\pi)=0$ and which derivative u' is square integrable on $(0,\pi)$. We shall consider on $\overset{\circ}{W}{}_2^1(0,\pi)$ the inner product

$$(u,v)_{1,2} = \int_0^\pi u'(x)\, v'(x)\, dx \quad .$$

For $u \in \overset{\circ}{W}{}_2^1(0,\pi)$ define $Lu \in \overset{\circ}{W}{}_2^1(0,\pi)$ such that

$$(Lu,v)_{1,2} = -\int_0^\pi u'(x)\, v'(x)\, dx + \int_0^\pi u(x)\, v(x)\, dx$$

for each $v \in \overset{\circ}{W}{}_2^1(0,\pi)$. Thus we have defined a linear continuous mapping $L: X \longrightarrow Z$ such that $Ker\,[L] =$ $=$ Linear hull $\{\sin x\}$.

Let $g : R_1 \longrightarrow R_1$ be a continuous function, $\sup_{\xi \in R_1} |g(\xi)| = M$. Obviously the mapping $N : X \longrightarrow Z$ defined by the relation

$$(Nu, v)_{1,2} = \int_0^{\pi} g(u(x))\, v(x)\, dx$$

for each $v \in \mathring{W}_2^1(0, \pi)$ is completely continuous and it satisfies the condition (B) from Section 3. Thus we have

Theorem 3. Suppose that there exists $\eta > 0$ such that

$$\lim_{x \to 0+} \frac{\min_{t \in \langle \eta, 1/x \rangle} g(t)}{x} > \frac{2M}{\pi} ,$$

$$\lim_{x \to 0+} \frac{\max_{t \in \langle -1/x, -\eta \rangle} g(t)}{x} < -\frac{2M}{\pi} .$$

Then for each $h \in L_2(0, \pi)$,

$$\int_0^{\pi} h(x) \sin x \, dx = 0 ,$$

there exists $u \in \mathring{W}_2^1(0, \pi)$ such that the integral identity

$$-\int_0^{\pi} u'(x) v'(x)\, dx + \int_0^{\pi} u(x) v(x)\, dx + \int_0^{\pi} g(u(x))\, v(x)\, dx = \\ = \int_0^{\pi} h(x)\, v(x)\, dx$$

holds for each $v \in \mathring{W}_2^1(0, \pi)$, i.e., the boundary value problem

$$\begin{cases} u'' + u + g(u) = h \\ u(0) = u(\pi) = 0 \end{cases}$$

is weakly solvable.

Remark 3. The same result is possible to give for general Sturm-Liouville operator of the second order and for the first eigenvalue of this operator.

b) <u>Boundary value problems for partial differential equations</u>.

The result from Section 3 it is not possible directly to apply to the weak solvability of boundary value problems for partial differential equations since we suppose that $X \subset L_\infty(\Omega)$ and X usually means the Sobolev space $\mathring{W}_2^1(\Omega)$. Thus we give the application to the classical solvability.

Let Ω be an open connected bounded subset of R_N with the boundary $\partial\Omega$. $C^k(\Omega)$ will denote the space of the functions which are k-times continuously differentiable on Ω and such that the derivatives can be extended continuously onto $\overline{\Omega}$. With the usual norm:

$$\|u\|_k = \sup_{0 \le |\kappa| \le k} \sup_{x \in \Omega} |D^\kappa u(x)|$$

$C^k(\overline{\Omega})$ is a Banach space, $C_0^k(\Omega)$ will denote the subspace of $C^k(\overline{\Omega})$ of the functions which are zero on $\partial\Omega$.

Finally we recall that the classical problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has countably infinite many eigenvalues $\{\lambda_n\}$, arranged according to increasing magnitude and considering their respective multiplicity. The last eigenvalue is simple: thus we have $0 < \lambda_1 < \lambda_2 \le \dots$. Moreover, the eigenfunction w_0 corresponding to λ_1 does not vanish on Ω and its values are of the same sign (this result is contained in a general theorem concerning the nodes of an eigenfunction - see e.g. [2, p.452]).

Set $X = C_0^2(\overline{\Omega})$, $Z = C^0(\overline{\Omega})$, $L: u \longmapsto \Delta u + \lambda_1 u$ $(L: C_0^2(\overline{\Omega}) \longrightarrow C^0(\overline{\Omega}))$. Since the condition (A) from Section 3 is fulfilled we have

Theorem 4. Let $g: R_1 \longrightarrow R_1$ be a continuous and bounded function satisfying the condition (C) from Section 3.

Then for each $h \in C^0(\overline{\Omega})$, $\int_\Omega h(x) w_0(x)\,dx = 0$ there exists a solution $u \in C_0^2(\overline{\Omega})$ of the problem

$$\begin{cases} \Delta u + \lambda_1 u + g(u) = h & \text{on } \Omega \\ \qquad u|_{\partial\Omega} = 0 . \end{cases}$$

Remark 4. The same result as in Theorem 4 it is possible to give for

$$L: u \longmapsto \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} a_i(x) \frac{\partial u}{\partial x_i} + a(x) u ,$$

where the coefficients a_{ij}, a_i, a are sufficiently smooth and L is uniformly elliptic operator.

References

[1] M.S. BERGER and M. SCHECHTER: On the solvability of semilinear operator equations and elliptic boundary value problems, Bull.Amer.Math.Soc.78(1972), 741-745.

[2] R. COURANT and D. HILBERT: Methods of Mathematical Physics, Vol.I, New York,1953.

[3] S. FUČÍK: Fredholm alternative for nonlinear operators in Banach spaces and its applications to diffe-

rential and integral equations, Čas.pěst.mat. 96(1971),371-390.

[4] S. FUČÍK: Nonlinear equations with noninvertible linear part, Czechoslovak Math.Journal (to appear)

[5] S. FUČÍK, M. KUČERA and J. NEČAS: Ranges of nonlinear asymptotically linear operators, J.Diff.Equations (to appear).

[6] S. FUČÍK, J. NEČAS, J. SOUČEK and V. SOUČEK: Spectral Analysis of Nonlinear Operators, Lecture Notes in Mathematics No 346,Springer Verlag 1973.

[7] P. HESS: On a theorem by Landesman and Lazer, Indiana Univ.Math.Journal(to appear).

[8] M.A. KRASNOSELSKIJ: Positive Solutions of Operator´s Equations (Russian), Moscow 1962.

[9] E.M. LANDESMAN and A.C. LAZER: Nonlinear perturbations of linear boundary value problems at resonance, Journ.Math.Mech.19(1970),609-623.

[10] J. NEČAS: Fredholm alternative for nonlinear operators with application to partial differential equations and integral equations, Čas.pěst.mat.97 (1972),65-71.

[11] J. NEČAS: On the range of nonlinear operators with linear asymptotes which are not invertible, Comment. Math.Univ.Carolinae 14(1973),63-72.

[12] L. NIRENBERG: An application of generalized degree to a class of nonlinear problems, Troisième Colloq. Anal.Fonct. Liège Centre Belge de Recherches Mathématiques, 1971, pp.57-73.

[13] L. NIRENBERG: Generalized degree and nonlinear problems, "Contributions to Nonlinear Analysis" edited by E. Zarantonello. Academic Press 1971, pp.1-9.

[14] M. SCHECHTER: A nonlinear elliptic boundary value problem (to appear).

[15] S.A. WILLIAMS: A sharp sufficient condition for solution of a nonlinear elliptic boundary value problem, J.Diff.Equations 8(1970),580-586.

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(Oblatum 31.1.1974)