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ON WEAK HOMOTOPY

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Abstract: If the definition of homotopy is weakened by using the cross-product instead of the usual cartesian product of spaces, all connected polyhedra become contractible.

Key-words: Cross-product, weak homotopy, polyhedron.

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The cross-product $X \otimes Y$ (the space obtained from the cartesian product of the underlying sets by the condition that $f: X \otimes Y \rightarrow Z$ is continuous iff it is continuous in each variable) is well-known to be a tensor product in the category of topological spaces. Thus, we can base on it a notion similar to homotopy - we will call it weak homotopy or W-homotopy - defined as follows:

$f, g: X \rightarrow Y$ are said to be W-homotopic if there is an $h: X \otimes I \rightarrow Y$ such that $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$.

Thus, W-homotopy is a weaker equivalence than the nor-

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mal one. In this paper we are going to show that it is actually much weaker: e.g. all connected polyhedra are W-homotopically trivial.

It is evident that every W-homotopically trivial space has to be arcwise connected. The converse is probably not true, but we do not have a counterexample. I am indebted to prof. Pultr, who suggested this problem, and who gave me valuable help.

1. Conventions and notations

Throughout this paper the circle is considered as the interval $[0,1]$, with identified endpoints. The closed (open) unit-interval will be denoted by $I(J)$. The closed unit-ball (sphere) in the n -dimensional Euclidean space R^n will be denoted by $B_n (S_n)$. The polyhedra will always be connected, and they are supposed to be embedded in a suitable Euclidean space. The points of this Euclidean space are sometimes considered as vectors - in order to simplify the notation. For every point $\rho \in R^n$, we define $U(\rho) = \vec{\rho} / \|\vec{\rho}\|$. Given two pointed spaces (X, x_0) and (Y, y_0) , $(X, x_0) \# (Y, y_0)$ is the topological space, obtained from $X \times Y$ identifying the points (x, y) with $x = x_0$ or $y = y_0$ (with the quotient-topology).

2.

Proposition 1. The products of W-homotopically trivial spaces are W-homotopically trivial.

Proof. Given a family $(X_\alpha)_\alpha$ of W-homotopically trivial spaces with homotopy-functions f_α , consider the following diagram:

$$\begin{array}{ccc}
 (\prod_{a \in A} X_a) \otimes I & \xrightarrow{\mu_B \otimes id_I} & X_B \otimes I \\
 \uparrow \prod_{a \in A} f_a & & \downarrow f_B \\
 (\prod_{a \in A} X_a) & \xrightarrow{\mu_B} & X_B
 \end{array}$$

where $\prod_{a \in A} f_a$ is defined in the following way:
 $\prod_{a \in A} f_a((x_a)_a, t) = (f_a(x_a, t))_a$. This function is continuous.

Proposition 2. The long line is W-homotopically trivial.

Proof. Let $L = \{(x, y) \mid x \in \mathbb{R}, y \in [0, 1[\}$ be endowed with the lexicographical order, and the associated order-topology. The function $h: L \otimes I \rightarrow L$; $h((x, y), t) = (xt, yt)$, is continuous, and L is W-homotopically trivial.

Proposition 3. The circle is W-homotopically trivial.

Proof. Consider $h: S \otimes I \rightarrow S$ defined by:

$$\begin{aligned}
 h(v, t) &= v^{1/t} & \text{if } t \neq 0 \\
 &= 0 & \text{if } t = 0.
 \end{aligned}$$

Clearly, h is continuous.

Corollary. Every torus is W-homotopically trivial.

3.

Suspension

Proposition. The suspension of an arbitrary space is W-homotopically trivial.

Proof. Let (X, x_0) be an arbitrary pointed space. Define $h: ((X, x_0) \times (S_1, 0)) \otimes I \rightarrow (X, x_0) \times (S_1, 0)$ by

$$h((x, \vartheta), t) = \begin{cases} (x, \vartheta^{1/t}) & \text{if } t \neq 0 \\ (x, 0) & \text{if } t = 0. \end{cases}$$

Let $g: (X, x_0) \times (S_1, 0) \rightarrow (X, x_0) \neq (S_1, 0)$ be the natural quotient-mapping. h is usually not continuous, but $g \circ h$ is. The commutativity of the diagram

$$\begin{array}{ccc} ((X, x_0) \times (S_1, 0)) \otimes I & \xrightarrow{g \circ h} & (X, x_0) \neq (S_1, 0) \\ \downarrow & & \nearrow h^* \\ ((X, x_0) \neq (S_1, 0)) \otimes I & & \end{array}$$

defines uniquely a continuous mapping h^* (because $g \otimes id$ is a quotient mapping).

Corollary. Every sphere is W -homotopically trivial.

4. Polyhedra

Proposition 1. All one-dimensional connected polyhedra are W -homotopically trivial. If x_0 is an arbitrary vertex of the polyhedron P , then the homotopy functions can be chosen in such a way that $\forall t \in I, f(x_0, t) = x_0$.

Proof. The proposition is trivial for all one-dimensional polyhedra with at most two vertices. Suppose it is proved for all one-dimensional polyhedra with at most $m-1$ vertices, $m \geq 3$. Let P be an arbitrary but fixed poly-

hedron with m vertices, embedded in a suitable \mathbb{R}^n , and suppose all segments of P have length 1. Choose an arbitrary vertex x_0 of P , denote the vertices of P by $(x_i)_{0 \leq i \leq m-1}$.

The segments $[x_i, x_j] \in P$, x_i and $x_j \neq x_0$, form at most $m-1$ maximal connected one-dimensional polyhedra P'_{k_i} ; $k_i \leq i_p \leq m-1$; $P'_{k_i} \cap P'_{k_i'} = \emptyset$ if $k_i \neq k_i'$. Choose $x_{j_{k_i}} \in P'_{k_i}$ such that $[x_{j_{k_i}}, x_0] \in P$, $\forall k_i \in i_p$. Consider the polyhedra P_{k_i} , consisting of the vertices of P'_{k_i} and x_0 , and all the segments in P between these vertices. By induction, the P'_{k_i} are W -homotopically trivial, and there exist continuous functions $f_{k_i}: P'_{k_i} \otimes I \rightarrow P'_{k_i}$ such that

$$f_{k_i}(x, 1) = x, \quad \forall x \in P'_{k_i}$$

$$f_{k_i}(x, 0) = x_{j_{k_i}}, \quad \forall x \in P'_{k_i}$$

$$f_{k_i}(x_{j_{k_i}}, t) = x_{j_{k_i}}, \quad \forall t \in I.$$

We will define the homotopy functions g_{k_i} on the polyhedra P_{k_i} . Suppose k_i fixed for the time being.

1) Consider the segment $[x_0, x_{j_{k_i}}]$.

Define $g_{k_i}(x, t) = t \cdot \overrightarrow{x_0 x}$ if $x \in [x_0, x_{j_{k_i}}]$.

2) Consider the polyhedron P'_{k_i} .

Define $d_{k_i}: P'_{k_i} \times P'_{k_i} \rightarrow \mathbb{R}_+$ by

$$d_{k_i}(y, y') = \inf \left\{ \sum_{a=1}^{m-1} \|x_a - x_{a-1}\| \mid x_1 = y, x_m = y', \right. \\ \left. x_a \in P'_{k_i}, [x_a, x_{a+1}] \subset P'_{k_i} \right\}.$$

A) $t = 1$

put $g_{j_{k_0}}(x, 1) = f_{j_{k_0}}(x, 1) = x$

B) $t \neq 1$

a) if $d_{j_{k_0}}(f_{j_{k_0}}(x, t), x_{j_{k_0}}) \geq 1/2$

put $g_{j_{k_0}}(x, t) = f_{j_{k_0}}(x, t)$

b) if $1/4 \leq d_{j_{k_0}}(f_{j_{k_0}}(x, t), x_{j_{k_0}}) \leq 1/2$

put $g_{j_{k_0}}(x, t) = 2(\overrightarrow{x_{j_{k_0}} f_{j_{k_0}}(x, t)} - (1/4) \cdot \overrightarrow{x_{j_{k_0}} x_{j_{k_0}}})$, where

$f_{j_{k_0}}(x, t) \in [x_{j_{k_0}}, x_j]$ and x_j is uniquely determined

c) if $0 \leq d_{j_{k_0}}(f_{j_{k_0}}(x, t), x_{j_{k_0}}) \leq 1/4$

put $g_{j_{k_0}}(x, t) = 4 \cdot d_{j_{k_0}}(f_{j_{k_0}}(x, t), x_{j_{k_0}}) \cdot \overrightarrow{f(x_{j_{k_0}}, t) x_{j_{k_0}}}$.

3) Consider the segments

$[x_0, x_j]$, $x_j \in P'_{j_{k_0}}$; $[x_0, x_j] \in P$, $j \neq j_{k_0}$.

Define $h_{j_{k_0}} : (P_{j_{k_0}} - P'_{j_{k_0}}) \otimes I \rightarrow R_+$ by

$h_{j_{k_0}}(x, t) = \|\overrightarrow{x_j x}\|^{1/t}$ if $t \neq 0$, and $x \in [x_0, x_j]$

0 if $t = 0$

if $t = 1$, put $g_{j_{k_0}}(x, 1) = x$

if $t \neq 1$

a) if $1/2 \leq h_{j_{k_0}}(x, t)$

put $g_{j_{k_0}}(x, t) = h_{j_{k_0}}(x, t) \cdot \overrightarrow{x_j x}$; $x \in [x_0, x_j]$

b) if $1/4 \leq h_{j_{k_0}}(x, t) \leq 1/2$

put $g_{j_{k_0}}(x, t) = 2(h_{j_{k_0}}(x, t) - 1/2) \cdot \overrightarrow{x_j x}$,

c) if $0 \leq h_{j_{k_0}}(x, t) \leq 1/4$,

i) $t = 0$

put $g_{j,k}(x, 0) = x_0$

ii) $t \neq 0$

put $q_{j,k} : J \rightarrow \mathbb{R}_+$ by

$$q_{j,k}(t) = \max \left\{ \sum_{i=1}^{m-1} \|g_{k}(x_j, t_i) - g_{k}(x_j, t_{i+1})\| \mid m \in \mathbb{N}, (t_i)_i \right.$$

partitions of $[t, 1]$, $[g_k(x_j, t_i), g_k(x_j, t_{i+1})] \subset P \}$.

Define $\kappa_{j,k} : \text{Im}(q_{j,k}) \rightarrow P_k$ by

$$\kappa_{j,k}(q_{j,k}(t)) = g_k(x_j, t),$$

define $\nu_{j,k,t} : [x_j, x_{t,j}] \rightarrow [0, q_{j,k}]$ by

$$\nu_{j,k,t}(x) = q_{j,k}(t) \cdot (1 - 4h_{k}(x, t)) \text{ if } x \in [x_j, x_0]$$

and where $x_{t,j}$ is that point on $[x_j, x_0]$ such that

$$h_{k}(x_{t,j}, t) = 1/4. \text{ Define } g_{j,k}(x, t) = \kappa_{j,k} \circ \nu_{j,k,t}(x)$$

if $x \in [x_j, x_0]$.

4) The polyhedron P . Define $g(x, t) = g_{j,k}(x, t)$

if $x \in P_k$. It is clear from the construction that $g :$

$P \otimes I \rightarrow P$ is a continuous function such that

$$g(-, 1) = id_P, \quad g(-, 0) = x_0.$$

Proposition 2. All connected polyhedra are W -homotopically trivial.

Proof. The theorem is proved for all one-dimensional polyhedra, suppose it is proved for all d -dimensional ones, with $d \leq m-1$, $m \geq 2$. Let P be an arbitrary fixed m -dimensional polyhedron embedded in a suitable \mathbb{R}^n . P' is the $(m-1)$ -dimensional skeleton of P , with a homotopy function g' .

A) Define $g(x, t) = g'(x, t)$ for $x \in P'$.

B) 1) There exist $f_{k'}: B_m \rightarrow K^{n'}$, $1 \leq k' \leq m$, such that

$$f_{k'}(B_m) \subset P, \quad \forall k' \leq m$$

$f_{k'}|_{B_m}$ is a homeomorphism onto the image

$$f_{k'}|_{S_n} \subset P'$$

$$f_{k'}(B_m) \cap f_{k''}(B_m) \subset P'; \quad k' \neq k''$$

$$\bigcup_{k'=1}^m f_{k'}(B_m) \cup P' = P.$$

2) If B_m is the unit-ball, define $h': B_m \times I \rightarrow B_m$ as follows:

a) $h'((0, 0, \dots, 0), t) = (1-t, 0, \dots, 0)$

b) $y \neq (0, 0, \dots, 0): h'(y, t) \in [(1-t, 0, \dots, 0), U(y)]$

and

$$\| \vec{y} \| = \frac{\| \vec{h}'(y, t) - (1-t, 0, \dots, 0) \|}{\| (1-t, 0, \dots, 0) - U(y) \|}$$

take an $h: B_m \otimes J \rightarrow B_m$ such that

$$h((0, 0, \dots, 0), t) = h'((0, 0, \dots, 0), t)$$

$$h(y, t) \in [(1-t, 0, \dots, 0), U(y)], \quad y \neq (0, 0, \dots, 0)$$

and

$$\frac{\| \vec{h}(y, t) - U(y) \|}{\| (1-t, 0, \dots, 0) - U(y) \|} = \left(\frac{\| \vec{h}'(y, t) - U(y) \|}{\| (1-t, 0, \dots, 0) - U(y) \|} \right)^{1/t}$$

3) If $x \in P - P'$, then $\exists! k' \leq m$ such that $x \in f_{k'}(B_m)$. Define the functions $h_{k'}: f_{k'}(B_m) \otimes J \rightarrow f_{k'}(B_m)$ by

$$h_{k_0}(x, t) = f_{k_0} \circ h((f_{k_0}^{-1}(x), t)) .$$

4) $x \in P'$.

Define $q_x : I \rightarrow R$ by

$$q_x(t) = \sup \left\{ \sum_{i=1}^{m-1} \| g^j(x, t_i) - g^j(x, t_{i+1}) \| \right\}$$

where $(t_i)_i$ are partitions of $[t, 1]$.

Define $x_x : \text{Im}(q_x) \rightarrow \text{Im}(g^j(x, -)) \subset P'$ by

$$x_x(q_x(t)) = g^j(x, t) .$$

5) $x \in f_{k_0}(B_m) - P'$; k_0 fixed.

a) Put $g_{k_0}(x, 1) = x$ and $g_{k_0}(x, 0) = x_0$, where $x_0 = g^j(-, 0)$

b) $t \in J$.

Notation:

$$v(y, t) = \| \overrightarrow{(1-t, 0, \dots, 0)} - \overrightarrow{U(y)} \|, \quad y \in B_m, \quad y \neq (0, 0, \dots, 0)$$

$$\mu_{k_0}(x, t) = d(h(f_{k_0}^{-1}(x), t), U(f_{k_0}^{-1}(x))) .$$

Let A_{x,t,k_0} and B_{x,t,k_0} be the points on the segment

$[(1-t, 0, \dots, 0), U(f_{k_0}^{-1}(x))]$ such that

$$\| \overrightarrow{A_{x,t,k_0}} - \overrightarrow{(1-t, 0, \dots, 0)} \| = v(f_{k_0}^{-1}(x), t) / 2$$

$$\| \overrightarrow{B_{x,t,k_0}} - \overrightarrow{(1-t, 0, \dots, 0)} \| = 3v(f_{k_0}^{-1}(x), t) / 4$$

1) If $v(f_{k_0}^{-1}(x), t) / 2 \leq \mu_{k_0}(x, t)$ put

$$g_{k_0}(x, t) = h_{k_0}(x, t) .$$

2) If $v(f_{k_0}^{-1}(x), t) / 4 \leq \mu_{k_0}(x, t) \leq v(f_{k_0}^{-1}(x), t) / 2$

define the linear functions

$$v_{x,t,h_e} : [A_{x,t,h_e}, B_{x,t,h_e}] \rightarrow [A_{x,t,h_e}, U(f_{h_e}^{-1}(x))]$$

such that

$$v_{x,t,h_e}(A_{x,t,h_e}) = A_{x,t,h_e}$$

$$v_{x,t,h_e}(B_{x,t,h_e}) = U(f_{h_e}^{-1}(x))$$

define $g_{h_e}(x,t) = f_{h_e} \circ v_{x,t,h_e} \circ h_e((f_{h_e}^{-1}(x), t))$.

$$3) \text{ If } 0 \leq \mu_{h_e}(x,t) \leq \nu(f_{h_e}^{-1}(x), t) / 4$$

define $\rho_{y,t,h_e} : [U(y), B_{x,t,h_e}] \rightarrow [0, \rho_x(t)]$, where

$x = f_{h_e}(y)$ and $x = f_{h_e}(U(y))$, to be the linear functions such that

$$\rho_{y,t,h_e}(B_{x,t,h_e}) = 0$$

$$\rho_{y,t,h_e}(U(y)) = \rho_x(t),$$

define $g_{h_e}(x,t) = x_x \circ \rho_{y,t,h_e}(x)$, where $x = f_{h_e}(y)$,
 $x = f_{h_e}(U(y))$.

$$4) \quad x \in P - P'$$

put $g(x,t) = g_{h_e}(x,t)$ if $x \in f_{h_e}(B_m)$.

The function $g : P \otimes I \rightarrow P$ is continuous.

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