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ON THE EXISTENCE OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR ELASTIC-INELASTIC SOLIDS (Preliminary communication)

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Abstract: This communication contains a brief discussion of the main result of the authors' paper: "On the solution of the traction boundary-value problem for elastic-inelastic materials", to appear in Archive for Rational Mechanics and Analysis.

Key words: Boundary-value problem, elastic-inelastic solids, internal variables, traction problem, constitutive equations, coercivity, convexity, contractive mapping.

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Introduction

Among attempts to formulate the theory of inelasticity adequate to real materials, a successful model of inelastic solids has been suggested (see e.g. [1 - 5]). The model is based on the concept of internal variables. Unlike the classical plasticity, the internal variable model comprises a wide range of mechanical properties: elastic, plastic and inelastic behavior, the rate and temperature dependence of stress-strain diagram, creep and relaxation effects.

In this note we briefly discuss internal variable model from the mathematical point of view. The formulation of a boundary-value problem and a sketch of the proof of the existence and uniqueness of solution for the internal variable model (for details of the proof see [9]) is illustrated here by the traction problem. The method of proofs of the existence and uniqueness of solutions given in [10, 11] is a modification of the method suggested in the paper [9].

Formulation of traction-boundary value problem

We consider a body with generic points x in the time interval $t \in \langle 0, T \rangle$. In traction boundary-value problem we look for the symmetric elastic strain tensor $\varepsilon_e = (\varepsilon^e_{ij})_{i,j=1,2,3}$, the symmetric inelastic strain tensor $\varepsilon_p = (\varepsilon^p_{ij})_{i,j=1,2,3}$, the symmetric stress tensor $\sigma = (\sigma_{ij})_{i,j=1,2,3}$, and the structural parameter vector $\alpha = (\alpha_i)_{i=1,\dots,m}$, such as ε_e, ε_p, $\sigma \in C(\langle 0,T\rangle, [L_2(\Omega)]^9)$, and $\alpha \in C(\langle 0,T\rangle, [L_2(\Omega)]^m)$.

We require that ε_e, ε_p, σ, and α satisfy the following conditions:

(i) The condition of compatibility

(1) $$\varepsilon_e + \varepsilon_p = \varepsilon , \quad \varepsilon \in X ,$$

where X is the linear modul of S defined as the set of $\varepsilon(v) = (\partial v_i / \partial x_j + \partial v_j / \partial x_i)/2$. By S we denote the subspace of $[L_2(\Omega)]^9$ of symmetric tensors with the scalar product

$$(\sigma, \tau) = \int_\Omega tr(\sigma\tau)\,dx .$$

v_i are from the Sobolev space $W_2^{(1)}(\Omega)$ of L_2 functions with L_2 first derivatives.

(ii) The equilibrium and boundary conditions

(2) $$\int_\Omega \mathrm{tr}\,(\sigma\, \varepsilon(v))\,dx - \int_\Omega F v\,dx - \int_{\partial\Omega} q v\,dS = 0$$

for every t from $\langle 0, T\rangle$ and every v from $[W_2^{(1)}(\Omega)]^3$.
The given body forces $F = (F_i)_{i=1,2,3}$ and surface tractions

$q = (q_i)_{i=1,2,3}$, $F \in C(\langle 0,T\rangle, [L_2(\Omega)]^3)$, $q \in C(\langle 0,T\rangle$, $[L_2(\partial\Omega)]^3)$, must satisfy the global equilibrium conditions

(3) $$\int_\Omega F\,dx + \int_{\partial\Omega} q\,dS = 0 \ ,$$

(4) $$\int_\Omega (F \times x)\,dx + \int_{\partial\Omega} (q \times x)\,dS = 0 \ .$$

(iii) The constitutive equations

(5) $$\varepsilon_e = A(\sigma, \alpha) \ ,$$

(6) $$\varepsilon_p(t) - \varepsilon_p(0) = \int_0^t B(\sigma(\tau), \alpha(\tau))\,d\tau \ ,$$

(7) $$\alpha(t) - \alpha(0) = \int_0^t C(\sigma(\tau), \alpha(\tau))\,d\tau \ .$$

We assume that the response functions A, B and C are lipschitz-like continuous and that there exists a function $P(\sigma, \alpha)$ with lipschitz-like continuous first derivatives with uniformly positively definite second differential in σ .

Existence and uniqueness of solution

In the paper [9] , the following theorem is proved:

Theorem. Under the conditions mentioned above there exists a unique solution of the traction boundary-value problem for the internal variable model.

Sketch of the proof: Let σ be in $C(\langle 0, \sigma \rangle, [L_2(\Omega)]^9)$ with σ enough small. It follows from (6) and (7) that the mappings $\sigma \longmapsto \varepsilon_p$ and $\sigma \longmapsto \alpha$ are contractions. Let us look for $\omega(t)$, $t \in \langle 0, \sigma \rangle$, which satisfies the equilibrium and boundary condition (2) and the compatibility relation (1) written in the form (see (1),(5) and the definitions of S and K)

$$(8)\quad \int_\Omega tr\left\{\left[\frac{\partial P}{\partial \sigma}(\omega, \alpha(\sigma)) + \varepsilon_p(\sigma)\right] h\right\} dx = 0 ,$$

where $h \in H$, $H \equiv S \dot{-} K$. But $\omega(t)$ is defined by (2) and (8) uniquely and we can find it by minimizing the functional

$$(9)\quad \int_\Omega [P(\omega, \alpha(\sigma)) + tr(\varepsilon_p(\sigma)\omega)]dx$$

in the space of functions satisfying (2). The conditions of coercivity and convexity for the functional (9) follow from the hypothesis. These conditions imply that the mapping $\sigma \longmapsto \omega$ is contractive, hence we obtain a unique fixed point. By a finite number of steps, replacing the interval $\langle 0, \delta \rangle$ by $\langle k\delta, (k+1)\delta \rangle$, we obtain the Theorem.

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