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LATTICE OF E-COMPACT TOPOLOGICAL SPACES

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Abstract: This paper is concerned with the following question: Let E, F be spaces such that $\mathcal{K}(E) \subsetneq \mathcal{K}(F)$, where $\mathcal{K}(P)$ is the class of all P-compact spaces; when there exists a space G such that $\mathcal{K}(E) \subsetneq \mathcal{K}(G) \subsetneq \mathcal{K}(F)$? A new class of atoms is found.

Key words: Epireflective subcategories, atoms, ordinals.

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E-compactness of a space was defined by Mrówka and Engelking in 1958. Let E be a topological space. A space P is said to be E-compact iff P is homeomorphic to a closed subspace of E^{m} for some suitable cardinal m. Let us denote a class of E-compact spaces by $\mathcal{K}(E)$. It holds for any non-discrete space E having more than one point: $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{K}(E)$, where $\mathcal{D}$ is a two-point discrete space. There is the following natural question:

(Q): Let E be a space, $\mathcal{D}$ a two-point discrete space (or more generaly: let E, F be spaces such that $\mathcal{K}(F) \subseteq \subseteq \mathcal{K}(E)$), when there exists a space G such that $\mathcal{K}(\mathcal{D}) \subsetneq \mathcal{K}(G) \subsetneq \mathcal{K}(E)$ ($\mathcal{K}(F) \subsetneq \mathcal{K}(G) \subsetneq \mathcal{K}(E)$ respectively).

If no such G exists, then we shall call E to be an atom (atom above F, respectively); this is a brief saying instead of " $\mathcal{K}(E)$ is an atom in the lattice of all classes of all P-compact spaces".

Mrówka discovered in [9] that the discrete space of natural numbers in an atom. The conjecture that the same situation occurs for any $T(\omega_\alpha)$ (ω_α is an initial ordinal) was based on this fact. However, in [1], Blefko constructed spaces P_α such that $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(P_\alpha) \subsetneqq \mathcal{K}(T(\omega_\alpha))$. It was not difficult to correct the proof and to generalize the construction - it will be introduced here as a construction M. In [2], Blefko published some results from [1] including his construction corrected in another way than our M - denote it for a moment by D. First, we supposed that both constructions M and D give homeomorphic spaces Z_α or A_α, respectively; but it is not the case: we shall prove that $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(A_\alpha) \subsetneqq \mathcal{K}(Z_\alpha) \subsetneqq \mathcal{K}(T(\omega_\alpha))$ for $\alpha \neq 0$. Blefko conjectured in [2] that there is no atom between $\mathcal{K}(\mathcal{D})$ and $\mathcal{K}(T(\omega_1))$, i.e. if $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(F) \subseteq \mathcal{K}(T(\omega_1))$, F is a space, then there exists a space G such that $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(G) \subsetneqq \mathcal{K}(F)$. But this conjecture was based on an example containing an obvious mistake. We will show that this conjecture is not true; the most general result in this direction which was achieved by us is Proposition 10 which solves some further special case of (Q). This paper is based on [11].

I wish to thank M. Hušek for his attention and valuable advice.

All spaces under considerations are supposed to be uniformizable Hausdorff; all spaces will be supposed to be nonvoid. A class of these spaces will be denoted by $\mathcal{T}$. Morphisms are continuous mappings. A set of morphisms from X into Y will be denoted by $C(X, Y)$.

First of all we introduce some well-known definitions and theorems which are necessary for the purposes of this paper.

<u>Definition</u>. Let X, E be spaces. X is said to be E-regular iff there exists a cardinal number m such that $X \subset E^m$. X is said to be E-compact iff there exists a cardinal number m such that $X \subset_{cl} E^m$.

A class of E-regular spaces will be denoted by $\mathcal{R}(E)$. A class of E-compact spaces will be denoted by $\mathcal{K}(E)$.

$\mathcal{K}(E)$ is epireflective in $\mathcal{T}$ and moreover if C is an epireflective subcategory in $\mathcal{T}$ containing E , then $\mathcal{K}(E) \subseteq C$ (it follows from Kenison's theorem).

<u>Notation</u>: Let $E \in \mathcal{T}$, $X \in \mathcal{T}$. Let $\kappa : X \longmapsto \kappa(X)$ be a $\mathcal{K}(E)$-reflection. A space $\kappa(X)$ will be denoted by $\beta_E X$, a morphism κ by β_E . It is easy to see that β_E is an embedding iff X is E-regular. We introduce some examples:

1) I is the closed interval $\langle 0, 1 \rangle$, $\mathcal{R}(I) = \mathcal{T}$, $\mathcal{K}(I)$ is the class of all compact spaces, β_I is Čech-Stone compactification. We shall denote it only by β .

2) R is the space of real numbers. $\mathcal{R}(R) = \mathcal{T}$, $\mathcal{K}(R)$ is the class of all realcompact spaces. β_R is Hewitt real-

compactification.

3) $\mathcal{D}$ is the two-point discrete space. $\mathcal{R}(\mathcal{D})$ is the class of all 0-dimensional spaces, $\mathcal{K}(\mathcal{D})$ is the class of all 0-dimensional compact spaces. (A space X is said to be 0-dimensional iff $ind\, X = 0$, i.e. X has a basis of clopen sets.)

4) N is the discrete space of natural numbers. $\mathcal{R}(N) = \mathcal{R}(\mathcal{D})$, $\mathcal{K}(N) \subsetneq \mathcal{K}(R) \cap \mathcal{R}(\mathcal{D})$. (As P. Nyikos proved, P. Roy's space is an element of $\mathcal{K}(R) \cap \mathcal{R}(\mathcal{D})$ but not of $\mathcal{K}(N)$, see [10].)

Convention: Symbols $I, R, \mathcal{D}, N$ will be used only in the sense as above. Let α be a limit ordinal number. $cf\,\alpha = min\,\{\beta \mid \alpha$ is cofinal with $\beta\}$.
For $E, X \in \mathcal{T}$ denote $f = \prod_{red}\{g \mid g \in C(X,E)\}$. Clearly $\beta_E X = \overline{f(X)}^{E^{card\, C(X,E)}}$. It follows immediately from this fact: if X is E-regular, then X is E-compact iff for each divergent net $\mathcal{N} = \{n_i\}_{i \in J}$ in X there exists $f \in C(X,E)$ such that $f \circ \mathcal{N}$ is divergent. It is proved in [12] that the assumption of E-regularity of X can be omitted.

Theorem (Blefko [1]). Let α, β be limit ordinals. $\mathcal{K}(T(\alpha)) = \mathcal{K}(T(\beta))$ iff $cf\,\alpha = cf\,\beta$.

This theorem shows that it is enough to consider only regular ordinals for solving the problem of atoms.

Proposition (Mrówka [9]): N is an atom.

<u>Theorem</u> (Mrówka [9]). Let E, F be spaces. Let $\mathcal{R}(E) = \mathcal{R}(F)$. Let X be E-regular space. Then $\beta_E X = \beta_F X - X_0$ where $X_0 = \{p_0 \in \beta_F X \mid$ there exists $Y \in \mathcal{K}(F)$ such that $E \subset Y$ and there exists $f \in C(\beta_F X, Y)$ such that $f(X) \subset E$ and $f(p_0) \in Y - E\}$.

A mapping f from a space X into a space Y is said to be perfect iff f is continuous, closed and $f^{-1}(y)$ is compact for each $y \in f(X)$. Let $X, Y \in \mathcal{R}(F)$. $f: X \to Y$ is said to be E-perfect iff f is continuous and $\tilde{f}(\beta_E X - X) \subseteq (\beta_E Y - Y)$, $\tilde{f}$ is an extension of f. This definition is a natural generalization of perfect mappings because any perfect mapping is just I-perfect (and, clearly, any perfect mapping is E-perfect for any E).

<u>Theorem.</u> Let X, E be spaces. Let Y be E-compact space. Let X be E-regular. If there exists an E-perfect mapping $f: X \to Y$, then X is E-compact.

<u>Proof</u>: X is E-regular, hence $\beta_E : X \to \beta_E X$ is an embedding. $f: X \to Y$ is E-perfect, hence $\beta_E \times f : X \to \beta_E X \times Y$ is an embedding on a closed subset of $\beta_E X \times Y$ which implies E-compactness of X.

<u>Convention</u>:1. We shall use for denoting of cardinality "omega" instead of "aleph". We hope it will be clear when ω_α denotes cardinality and when ω_α is used as an initial ordinal number.

2. If we say that a space P has locally some property H, we suppose that each point of P possesses a basis of

neighbourhoods consisting of members with a property $\mathcal{H}$.

Construction $\mathcal{M}$:

Definition 1. Let P be a non-discrete locally compact locally sequentially compact space. Put $P_b = \{x \in P \mid$ there exists a non-constant sequence $\{x_n\}_{n=1}^{\infty}$ in P converging to $x\}$ and suppose that for each point $x \in P_b$ there exists its neighbourhood U_x such that $U_x - \{x\}$ is normal. Define a set $Z(P) = \bigcup_{x \in P - P_b} \{\langle x, x \rangle\} \cup \bigcup_{x \in P_b} \{x\} \times (\beta(\omega P - \{x\}) - (\omega P - x)$ where ωP is the Alexandrov one-point compactification of P . Define for each $x \in P$ a space P_x and a mapping $f_x : Z(P) \longrightarrow P_x$. Put $P_x = P$ for $x \in P - P_b$, $P_x = \beta(\omega P - \{x\}) - (\omega P - P)$ for $x \in P_b$. If $x \in P - P_b$, then $f_x(\langle y, y \rangle) = y$ for $y \in P$, $f_x(\langle y, z \rangle) = y$ for $y \in P_b$, $z \in \beta(\omega P - \{y\}) - (\omega P - \{y\})$ .If $x \in P_b$, then $f_x(\langle y, z \rangle) = y$ for $\langle y, z \rangle \in Z(P)$, $y \neq x$, $f_x(\langle x, z \rangle) = z$ for $\langle x, z \rangle \in Z(P)$. Furnish a set $Z(P)$ by a topology $\mathcal{U}$ projectively generated by $\{f_x \mid x \in P\}$.

Remarks: 1) Definition 1 is correct because it is easy to see that $\langle Z(P), \mathcal{U} \rangle$ is a uniformizable Hausdorff space (P_x is uniformizable for each $x \in P$).

2. Speaking about the space $Z(P)$, we shall have always in mind a space $\langle Z(P). \mathcal{U} \rangle$ just defined.

3) Obviously, we could use in Definition 1 other sorts of compactifications instead of β . It will be clear that it could simplify sometimes our situation.

Definition 2. Let P satisfy the conditions of Definition 1. Let $Z(P)$ be a space defined in Definition 1. A mapping $p: Z(P) \longrightarrow P$ is defined in the following way: $p(\langle y, x \rangle) = y$ for $\langle y, x \rangle \in Z(P)$.

<u>Proposition 1.</u> p from Definition 2 is perfect.

<u>Proof</u>: 1) p is continuous. Choose $x \in P$ and a neighbourhood U of x . If there exists $x_0 \in P - P_b$, then $f_{x_0}^{-1}(U) = p^{-1}(U)$, hence $p^{-1}(U)$ is a neighbourhood of a set $p^{-1}(x)$. If $P = P_b$ choose some $x_1 \in P$, $x \neq x_1$. $f_{x_1} : Z(P) \longrightarrow P_{x_1}$ is continuous. There exists the only mapping $\varphi_{x_1} : P_{x_1} \longrightarrow P$ such that $p = \varphi_{x_1} \circ f_{x_1}$. It is easy to see that φ_{x_1} is continuous and $\varphi_{x_1}(P_{x_1} - \varphi_{x_1}^{-1}(x_1))$ is even a homeomorphism. It implies immediately that $p^{-1}U$ is a neighbourhood of the set $p^{-1}(x)$.

2) For each $x \in P$, $p^{-1}(x)$ is compact. Either $x \in P - P_b$, then $p^{-1}(x) = \langle x, x \rangle$, or $x \in P_b$, then $p^{-1}(x) = \beta(\omega P - \{x\}) - (P - \{x\})$, hence the preimage of any point is compact.

3) p is closed. The following assertion will be used: A mapping $g : A \longrightarrow B$ is closed iff for each neighbourhood U of the set $g^{-1}(x)$, $x \in B$, there exists a neighbourhood V of x such that $g^{-1}(V) \subset U$.

Let $x \in P$. Choose some neighbourhood U of the set $p^{-1}(x)$. One can assume that $U = f_{x_0}^{-1}(O)$ where $x_0 \in P$, O is a neighbourhood of a set $f_{x_0}(p^{-1}(x))$. If $x \neq x_0$, then $O - f_{x_0}(p^{-1}(x_0))$ is a neighbourhood of $f_{x_0}(p^{-1}(x))$ and it is enough to find out that

$\varphi_{x_0} / P_{x_0} - \mathcal{E}_{x_0}(p^{-1}(x_0))$ is a homeomorphism. Let $x = x_0$ now. If we find a neighbourhood V of x_0 such that $\varphi_{x_0}^{-1}(V) \subset O$, then the proof is finished as $p^{-1}(V) = \mathcal{E}_{x_0}^{-1}(\varphi_{x_0}^{-1}(V)) \subset \mathcal{E}_{x_0}^{-1}(O) = \mathcal{U}$. Suppose the contrary: For each neighbourhood W of x_0, there exists $x_W \in W$ such that $\varphi_{x_0}^{-1}(x_W) \notin O$ (it implies $x \neq x_0$). Choose some compact neighbourhood O_{x_0} of x_0. Put $\mathcal{W} = \{W \mid W$ is a neighbourhood of x_0 and $W \subset O_{x_0}\}$. The net $\mathcal{W} = \{x_W\}_{W \in \mathcal{W}}$ converges to x_0. $\varphi_{x_0}^{-1} \circ \mathcal{N} = \{\varphi_{x_0}^{-1}(x_W)\}_{W \in \mathcal{W}}$ is a net in the compact space $\beta(O_{x_0} - \{x_0\})$. There exists a subnet $\mathcal{M}$ of $\varphi_{x_0}^{-1} \circ \mathcal{N}$ converging to some $x_1 \in \beta(O_{x_0} - \{x_0\})$. Clearly $x_1 \in \beta(O_{x_0} - \{x_0\}) - (O_{x_0} - \{x_0\})$, i.e. $\varphi_{x_0} x_1 = x_0$. However φ_{x_0} is continuous, hence $\varphi_{x_0} \circ \mathcal{M}$ converges to $\varphi_{x_0}(x_1)$ which is a contradiction with properties of $\mathcal{N}$ and of Hausdorff spaces.

<u>Remark</u>: No special properties of Čech-Stone compactification have been exploited in the proof of Proposition 1.

<u>Corollary 1</u>. If $Z(P)$ is P-regular, then $Z(P)$ is P-compact. (In particular, if both P and $Z(P)$ are 0-dimensional, then $Z(P)$ is P-compact.)

<u>Corollary 2</u>. $Z(P)$ is locally compact. P is compact iff $Z(P)$ is compact.

<u>Lemma 1</u>. Let P be a space satisfying the conditions

of Definition 1, $Z(P)$ a space defined in the same definition, p the mapping from Definition 2. If $\{a_m\}_{m\in N}$ is a converging sequence in $Z(P)$, then $card\ p\{a_m \mid m\in N\} < \omega_0$.

Proof: Proof follows immediately from the fact that there exists no sequence $\{b_m\}_{m\in N}$ in $\omega P-\{x\}$, $x\in P_b$ which converges to a point of $\beta(\omega P-\{x\})-(\omega P-\{x\})$.

Remark: We must employ also the fact that for each $x\in P_b$, there exists its neighbourhood U such that $U-\{x\}$ is normal and properties of βN.

Proposition 2. Let Q be a sequentially compact space. Let P be a space satisfying the conditions of Definition 1. If $f: Q\longrightarrow Z(P)$ is continuous, then $card\ pf(Q) < \omega_0$.

Proof: Suppose $card\ pf(Q)\ge\omega_0$. Q is the sequentially compact space, hence there exists a sequence $\{x_m\}_{m\in N}$ in Q such that $(i,j\in N, i\neq j\Rightarrow pfx_i\neq pfx_j)$ and $\{x_m\}$ converges to $x_0\in Q$. f, p are continuous, consequently it contradicts Lemma 1.

Corollary 1. If Q is sequentially compact and non-compact, then $Q\notin\mathcal{K}(Z(P))$.

Corollary 2. If Q is a locally sequentially compact connected space and $f: Q\longrightarrow Z(P)$ is continuous, then $card\ pf(Q) < \omega_0$.

Proof: Proof follows from the fact that for any point

$q \in Q$ there exists its neighbourhood $\mathcal{U}$ such that $card\ pf(\mathcal{U}) \doteq 1$.

Corollary of Corollary 2: If Q is a locally sequentially compact connected non-compact space, then $Q \notin \mathcal{K}(Z(P))$.

Proposition 3. Let P satisfy the conditions of Definition 1. Let Z(P) be P-regular (it holds, when both P and Z(P) are 0-dimensional). Then $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(Z(P)) \subsetneqq \mathcal{K}(P)$ whenever one of the following conditions is satisfied:
1) P is sequentially compact and non-compact, 2) P is locally sequentially compact connected non-compact.

Proof: It follows from Corollary 2 of Proposition 1 and from Proposition 2 and its corollaries.

Application. Let ω_α be a regular initial ordinal number, $\alpha \neq 0$. (It was mentioned above that the case of singular ordinals is not so interesting, moreover, it could be shown that if $cf\,\omega_\alpha = cf\,\omega_\beta$, then $\mathcal{K}(Z(T(\omega_\alpha)) = \mathcal{K}(Z(T(\omega_\alpha))$.) The space $T(\omega_\alpha)$ satisfies the conditions of Definition 1. The space $Z(T(\omega_\alpha))$ will be denoted merely by Z_α . $T(\omega_\alpha)$ is sequentially compact and non-compact. It means: for proving the fact that $\mathcal{K}(\mathcal{D}) \subsetneqq \mathcal{K}(Z_\alpha) \subsetneqq \mathcal{K}(T(\omega_\alpha))$, it remains to show that Z_α is 0-dimensional (see Proposition 3). But it follows from the following two propositions and Definition 1.

Proposition 4. Let X be a space. Then $ind\ \beta X = 0$ iff $Ind\ X = 0$.

Proposition 5. Let X be a generalized ordered space. Then $ind\, X = 0$ iff $Ind\, X = 0$.

We shall not prove these propositions, the proof of the first one is well-known (see e.g. [4]), the proof of the second one is similar to the proof of normality of an ordered space.

Remarks. 1) If we use Banaschewski compactification (i.e. $\beta_{\mathfrak{D}}$) in Definition 1, we need not Propositions 4 and 5 and the constructed space for 0-dimensional P must be 0-dimensional. (For Lemma 1 consider that $\beta N = \beta_{\mathfrak{D}} N$.)

2) It might be interesting for someone that Z_{α}, $\alpha \neq 0$, has the one-point Čech-Stone compactification. It holds more generally: If P satisfies the conditions of Definition 1, P is countably compact and has the one-point Čech-Stone compactification, then $Z(P)$ has also the one-point Čech-Stone compactification.

Lemma 2. Let K be a compact space. Let $x \in K$. If $K - \{x\}$ is realcompact, then every infinite closed subset of $\beta(K - \{x\}) - (K - \{x\})$ contains a copy of βN .

Remark: If $x \in K$ is a G_{δ}-point then $K - \{x\}$ is realcompact.

Proof: See [4].

Proposition 6. Let P satisfy the conditions of Definition 1 and, in addition, the following one: there exists for any point $x \in P_{b}$ its neighbourhood U_x such that $U_x - \{x\}$ is realcompact. Then there are no converging sequences in $Z(P)$.

Proof: Easy. (Compare with Lemma 1.)

Corollary. Let P be a space satisfying the conditions from Proposition 6. Let Q be a space. $Q \notin \mathcal{K}(Z(P))$ whenever one of the following two conditions is satisfied:
1) Q is sequentially compact and it is not 0-dimensional; 2) Q is locally sequentially compact space containing an infinite connected subspace S.

Proof: Similar as in Proposition 2 and its corollaries.

Application.

$$\begin{array}{ccccc} \mathcal{K}(\mathcal{D}) & \subsetneq & \mathcal{K}(Z(I)) & \subsetneq & \mathcal{K}(I) \\ & & \neq\ \cap & & \neq\ \cap \\ & & \mathcal{K}(Z(R)) & \subsetneq & \mathcal{K}(R) \\ & & \neq\ \cap & & \parallel \\ & & \mathcal{K}(Z(R^2)) & \subsetneq & \mathcal{K}(R^2) \end{array}$$

Remark: We do not know whether $\mathcal{K}(Z(R^j)) \subsetneq \mathcal{K}(Z(R^{j+1}))$ for $j > 1$.

Note: Let ω_α be a regular ordinal, $\alpha \neq 0$. One can construct a space, call it P_α, such that $\mathcal{K}(P_\alpha) = \mathcal{K}(Z_\alpha)$ but P_α is not homeomorphic to Z_α. The main reason why we are going to introduce this new construction is a possibility of a generalization of Construction $\mathcal{M}$ using this new one.

Put $S = \{\gamma \mid \gamma < \omega_\alpha,\ cf\,\gamma = \omega_0\}$.

Choose for each $x \in S$ a strictly increasing sequence $s_x = \{x_n\}_{n \in N}$ such that s_x converges to x and each member of

s_x is an isolated ordinal. Put $U_x^n = \{y \mid x_n \le y < x_{n+1}\}$. Define a set $P_\alpha = \bigcup_{x \in T(\omega_\alpha) - S} \{\langle x, x\rangle\} \cup \bigcup_{x \in S} \{x\} \times (\beta N - N)$. We shall define a mapping $p: P_\alpha \to T(\omega_\alpha)$: $p(\langle x, y\rangle) = x$ for $\langle x, y\rangle \in P_\alpha$. Define a topology on P_α: take as a basis of neighbourhoods of $x \in p^{-1}(T(\omega_\alpha) - S)$ a family $\{p^{-1}(U) \mid U$ is a neighbourhood of $p(x)\}$. A family $I^{\langle x,y\rangle} = \{\bigcup_{j \in O \cap N} U_x^j \cup$ $\cup (O \cap (\beta N - N)) \mid O$ is a neighbourhood of $y \in (\beta N - N)$ in $\beta N\}$ is a basis of neighbourhoods of $\langle x, y\rangle \in p^{-1}(S)$. The topology will be the same if we define $M^{\langle x,y\rangle} = \{B \mid B \supset U$, $U \in I^{\langle x,y\rangle}$, $p(B)$ is a neighbourhood of $p(U)\}$ as a basis of neighbourhoods of $\langle x, y\rangle \in p^{-1}(S)$. Clearly P_α with this topology is a space. It is easy to see that P_α has really the promised properties.

After a little modification of the construction, we can apply this construction e.g. to spaces with the first countability axiom and these spaces need not be 0-dimensional. Take a basis of a point x. Denote this basis by B_x. One can suppose that $B_x = \{K_n\}_{n \in N}$ and $K_n \supset K_{n+1}$. Put $U_n^x = K_n - K_{n+1}$. Define a family $M^{\langle x,y\rangle}$ as a basis of neighbourhoods of $\langle x, y\rangle$.

Clearly, we can join Construction M and the new one in a construction which generalize both of them (i.e. the last construction can be applied to a larger class of spaces and the question Q might be answered in more general cases).

II. Atoms

Notation: $D(\omega_\alpha)$ is a discrete space of cardinality ω_α. $\beta_{T(\omega_\alpha)} D(\omega_\alpha)$ will be denoted by A_α.

Definition 3.x) Let P be a space, ω_α an initial ordinal. P is said to have the property U_α iff each subset of P of cardinality less than ω_α has a compact closure.

Remarks: 1) Clearly, the class of all spaces with the property U_α is an epireflective subcategory in $\mathcal{T}$.

2) See [13] for the property U_1.

Definition 4. Let P be a space. Let $\mathcal{N} = \{n_j\}_{j \in J}$ be a net in P. $\mathcal{N}$ is said to be an α-net (α is an ordinal) iff there exists $j_0 \in J$ such that $\operatorname{card}\{x \mid x \in P \,\&\, \exists i \in J : i \geq j \,\&\, x = n_i\} = \omega_\alpha$ for each $j \in J$, $j > j_0$.

Remark: For each ω_α there exists a space containing an α-net any of its subnets is also an α-net (take $\beta D(\omega_\alpha)$). Such a net is said to be a regular α-net.

Definition 5. Let P be a space, ω_α and initial ordinal. P has the property K_α iff: 1) P has the property U_α.

2) For each regular α-net $\mathcal{N}$ in $D(\omega_\alpha)$ there exists $f : D(\omega_\alpha) \longrightarrow P$ such that each subnet of $f \circ \mathcal{N}$ diverges.

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x) After finishing this paper I found out that spaces with the property U_α are called α-bounded in [14].

Remark: We do not know whether P having the property U_α, but not $U_{\alpha+1}$, has to have the property K_α. We do not know whether there is a space with $U_\alpha(K_\alpha)$ but not with $U_{\alpha+1}(K_{\alpha+1})$ for ω_α singular.

Proposition 7. Let ω_α be a regular initial ordinal. The space $T(\omega_\alpha)$ has the property K_α.

Proof is obvious.

Proposition 8. Let ω_α be a regular initial ordinal. $A_\alpha = \beta D(\omega_\alpha) - Y_0$ where $Y_0 = \{x \in \beta(D(\omega_\alpha)) \mid$ there exists a regular α-net of points of $D(\omega_\alpha)$ converging to $x\}$. (Obviously $Y_0 = \{x \in \beta D(\omega_\alpha) \mid$ (for all $A \subset D(\omega_\alpha)$: $: x \in \overline{A}^{\beta D(\omega_\alpha)} \Longrightarrow \mathrm{card}\, A = \omega_\alpha\}$.

Proof: Let $f: D(\omega_\alpha) \longrightarrow T(\omega_\alpha)$ be a bijection, $\tilde{f}$: $: \beta(D(\omega_\alpha)) \longrightarrow T(\omega_\alpha + 1)$ the Stone extension of f. There exists no α-net converging in $T(\omega_\alpha)$, hence $\tilde{f}(Y_0) = \{\omega_\alpha\}$. It holds further: if $\mathcal{N}$ is a net in $T(\omega_\alpha)$ converging to ω_α in $T(\omega_\alpha + 1)$, then $\mathcal{N}$ is an α-net. It means that $g^{-1}(\omega_\alpha) \subset Y_0$ for each $g \in C(\beta D(\omega_\alpha), T(\omega_\alpha + 1))$. Now the proposition follows from Mrówka's theorem. Proof for $\alpha = 0$ is selfevident.

Proposition 9. Let ω_α be a regular initial ordinal. Let P be a space with the property K_α. Then A_α is P-compact.

Proof: The characterization of E-compactness using a concept of nets shall be employed. Let $\mathcal{N}$ be a net in A_α

no subnet of which converges in A_α . There exists a subnet $\mathfrak{M}$ of $\mathfrak{N}$ such that $\mathfrak{M}$ converges in $\beta D(\omega_\alpha)$ to $d \in \beta D(\omega_\alpha)$. Necessarily: $d \in Y_0$ (see Proposition 8). Hence there exists a reguler α -net $\mathcal{S}$ in $D(\omega_\alpha)$ converging to d. P has the property K_α , consequently there exists $f : D(\omega_\alpha) \longrightarrow P$ such that each subnet of $f \circ \mathcal{S}$ diverges. Consider $\tilde{f}: \beta D(\omega_\alpha) \longrightarrow P$ ($\tilde{f}$ is the Stone extension of f). $A_\alpha = \beta D(\omega_\alpha) - Y_0$, P has the property U_α , $\tilde{f}$ is continuous, hence $\tilde{f}(A) \subset P$. $\tilde{f}(d))$ must be an element of $\beta P - P$: $\mathcal{S}$ converges to $d \in$ $\in \beta D(\omega_\alpha)$, hence $\tilde{f} \circ \mathcal{S}$ converges to $\tilde{f}(d)$ in βP ; if $\tilde{f}(d) \in P$, $f \circ \mathcal{S}$ would converge in P which is impossible, Let us denote $g = \tilde{f}/A$. Then a net $g \circ \mathfrak{N}$ diverges in P: if $g \circ \mathfrak{N} \longrightarrow q \in P$, then also $g \circ \mathfrak{M} \rightarrow q$ P , but $g \circ \mathfrak{M} = f \circ \mathfrak{M}$ converges to $\tilde{f}(d)$ in βP .

<u>Lemma 3.</u> Let P be a space having the property U_α (ω_α is a regular initial ordinal). If there exists a continuous mapping $f : P \longrightarrow T(\omega_\alpha)$ such that $\operatorname{card} f(P) = \omega_\alpha$, then P has the property K_α .

<u>Proof</u> is obvious.

<u>Corollary</u>. Any $T(\omega_\alpha)$ -compact space (ω_α is regular) that is not compact has the property K_α .

<u>Corollary</u> of Corollary: $cf(\omega_\alpha) \neq cf(\omega_\gamma) \Longrightarrow (\mathcal{K}(T(\omega_\alpha)) \cap \mathcal{K}(T(\omega_\gamma)) = \mathcal{K}(\mathcal{D}))$. Hence $cf\alpha \neq cf\beta$ iff $\mathcal{K}(T(\alpha)) \cap \mathcal{K}(T(\beta)) = \mathcal{K}(\mathcal{D})$.

Theorem 1. Let ω_α be a regular initial ordinal. A_α is an atom.

Proof: Let E be such a space that $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{K}(E) \subseteq \mathcal{K}(A_\alpha)$. If $\mathcal{K}(\mathcal{D}) \neq \mathcal{K}(E)$, then E has the property K_α, hence $\mathcal{K}(A_\alpha) = \mathcal{K}(E)$.

Remark: We do not know how the assumption of regularity of ω_α is important.

Definition 6. Let X, P be spaces. X is said to be an atom above P iff: 1) $\mathcal{K}(P) \subsetneq \mathcal{K}(X)$, 2) $(\mathcal{K}(P) \subseteq \mathcal{K}(Q) \subseteq \mathcal{K}(X)) \Longrightarrow (\mathcal{K}(P) = \mathcal{K}(Q)$ or $\mathcal{K}(Q) = \mathcal{K}(X)$.

Proposition 10. Let ω_α be an initial ordinal, ω_σ a regular initial ordinal. Let $\omega_\alpha > \omega_\sigma$. Let P be a space with the property $\mathcal{U}_\alpha$ and $\operatorname{card} P \geq 2$. Then $P \times A_\sigma$ is an atom above P.

Proof: A_α does not have the property $\mathcal{U}_\alpha$, hence $\mathcal{K}(P) \subsetneq \mathcal{K}(P \times A_\sigma)$. Let Q be a space such that $\mathcal{K}(P) \subseteq \mathcal{K}(Q) \subseteq \mathcal{K}(P \times A_\sigma)$, $a: P \times A_\sigma \longrightarrow A_\sigma$ is a projection. If $\overline{a \circ g(Q)}$ is compact for each $g \in C(Q, P \times A_\sigma)$, then $\mathcal{K}(Q) = \mathcal{K}(P)$ because $\mathcal{K}(\mathcal{D}) \subseteq \mathcal{K}(P)$. If $\overline{a \circ g(Q)}$ is not compact for some $g \in C(Q, P \times A_\sigma)$, then Q has the property K_σ, hence $\mathcal{K}(Q) = \mathcal{K}(P \times A_\sigma)$.

Remarks: 1) We do not know whether for each space E with $\mathcal{K}(\mathcal{D}) \subsetneq \mathcal{K}(E)$ there exists an atom such that $\mathcal{K}(A) \subseteq \mathcal{K}(E)$.

2) Obviously: If A is an atom, then $\mathcal{K}(A) = \mathcal{K}(\beta(\omega_\alpha) - X)$

for suitable ω_α and $X \subseteq \beta D(\omega_\alpha)$.

III Relation between A_α and Z_α for regular ω_α :

Lemma 4. Let P be a space satisfying the conditions of Definition 1. Let $(J, \leq)$ be a right-directed set. Let $\mathfrak{N} = \{n_i\}_{i\in J}$, $\mathfrak{M} = \{m_i\}_{i\in J}$ be nets in $Z(P)$ such that 1) $p(n_i) = p(m_i)$ for each $i \in J$, p is the mapping from Definition 2; 2) $\mathfrak{N}$ converges to x in $Z(P)$; 3) $n_i \notin p^{-1}(p(x))$ for each $i \in J$. Then $\mathfrak{M}$ converges to x .

Proof follows immediately from the definition of the topology on $Z(P)$.

Proposition 11. $\mathcal{K}(A_\alpha) \subsetneqq \mathcal{K}(Z_\alpha)$ for regular ω_α, $\alpha \neq 0$.

Proof: Z_α has the property K_α i.e. it holds $\mathcal{K}(A_\alpha) \subseteq \mathcal{K}(Z_\alpha)$. It remains to prove $Z_\alpha \notin \mathcal{K}(A_\alpha)$. Suppose the contrary: Then there exists $f \in C(Z_\alpha, A)$ such that $\overline{f(Z_\alpha)}$ is not compact. Put $J^\gamma = \{\delta \in T(\omega_\alpha) \mid \delta > \gamma$, δ is an isolated ordinal$\}$ for $\gamma \in T(\omega_\alpha)$. If $\operatorname{card} f(p^{-1}(J^\delta)) < \omega_0$ for some $\delta \in T(\omega_\alpha)$, then $\overline{f(Z_\alpha)}$ would be compact (p is the mapping from Definition 2). Using properties of βN one can easily prove that there exist sets A, B such that $A \subset Z_\alpha, B \subset Z_\alpha, f(A)$ and $f(B)$ are mutually disjoint countable isolated sets and $\overline{A} \cap \overline{B} \neq \emptyset$ - a clear contradiction $(\overline{f(A)} \cap \overline{f(B)} = \emptyset)$.

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