Commentationes Mathematicae Universitatis Carolinae

Manh Quy Nguyen

g-monomorphisms

Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 4, 669--687

Persistent URL: http://dml.cz/dmlcz/105518

Commentationes Mathematicae Universitatis Carolinae

14,4 (1973)

g - MONOMORPHISMS x)

NGUYEN MANH QUY, Praha

Abstract: The aim of this paper is to propose a "good" definition of subobjects.

Key words: Subobject, g-monomorphism, concrete category.

AMS: 18A20 Ref. Ž. 2.726.1

A categorial definition of subobjects should be satisfactory from two points of view: First, in categories induced by current structures it should characterize (some of) the structurally defined subobjects. Second, it should satisfy some natural requirements such as preservation under compositions, intersections, pullbacks etc.

In some categories, e.g. primitive classes of algebras, category of sets, category of compact Hausdorff spaces, one can represent subobjects simply by all monomorphisms. In general, giving a definition of subobjects means to determine some particular ones among the monomorphisms.

There have been many concepts proposed, e.g. equalizer, extremal monomorphism [8], strong monomorphism [6]. These already represent e.g. subspaces in the category of topolo-

----

x) This is a part of my thesis.

gical spaces, full subgraphs in the category of graphs etc. On the other hand, in the category of Hausdorff spaces it characterizes the closed subspaces. The first two are not closed under compositions even in a category with somewhat "nice" properties x). The copure monomorphisms [5] characterize all subspaces in the category of Hausdorff spaces; they sometimes fail to compose xx).

In this paper we present a definition of g-monomorphisms and study their properties and then describe them in concrete categories, especially in the categories structurally defined. We show that this notion suits well for describing subobjects.

I am indebted to A. Pultr and M. Hušek for valuable advice and particularly for guiding in my study.

----

x) For the extremal monomorphisms, see the example in [1] (in the section 8, Appendix). For the equalizers, at first see the example in § 2 of [6] which shows that a composition of two regular monomorphisms may not be a regular monomorphism even in a category complete, cocomplete and additive. Then, observe that in a complete category a regular monomorphism is nothing other than an equalizer of a pair. For this, prove at first that

1) Equalizers in a category having products are closed under compositions.

2) The simultaneous equalizer of a family of pairs $f_i, g_i : A \longrightarrow B_i$ (see § 1 [6]) is, in a category having equalizers, the intersection of the family of the equalizers of the pairs $f_i, g_i, i \in I$.

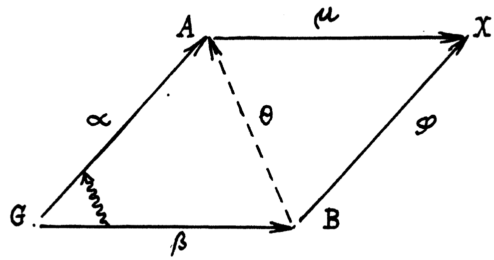
Consequently, in a complete category simultaneous equalizers and equalizers coincide. At last, observe that the simultaneous equalizers and the regular monomorphisms are the same.

xx) In [1] P. Arduini has given an example which shows that a composition of two extremal monomorphisms may not be an extremal monomorphism even in a canonical category. This is the same example for copure monomorphisms.

§ 1. Definitions

Definition 1. A monomorphism $\mu : A \longrightarrow X$ in a category $\mathcal{A}$ is called a g-monomorphism if for every morphism $\varphi : B \longrightarrow X$ the following implication holds:

if there is a generator G of $\mathcal{A}$ such that for every $\beta : G \longrightarrow B$ there is an $\alpha : G \longrightarrow A$ with $\mu\alpha = \varphi\beta$, then there is a morphism $\theta : B \longrightarrow A$ such that $\varphi = \mu . \theta$.



Definition 2. Let G be a generator of $\mathcal{A}$. A monomorphism $\mu : A \longrightarrow X$ in $\mathcal{A}$ is called a G-injection if for every morphism $\varphi : B \longrightarrow X$ the following implication holds:

if for every morphism $\beta : G \longrightarrow B$ there is an $\alpha : G \longrightarrow A$ with $\mu\alpha = \varphi\beta$, then there is a morphism $\theta : B \longrightarrow A$ with $\varphi = \mu . \theta$.

Thus, μ is a g-monomorphism if and only if it is a G-injection for any generator G.

Definition 3. A generator G in a category $\mathcal{A}$ is said to be absolute if it is a retract of any generator of $\mathcal{A}$.

For example, a single-point set is an absolute generator in the category of sets, a single-point space is an absolute generator in the category of topological spaces. More generally, we have the following proposition.

Proposition 1. Let $\mathcal{A}$ be a category which is not thin x). Let G be a generator of $\mathcal{A}$ such that $\mathcal{A}(G,G)=\{1_G\}$. Then G is an absolute generator.

Proof. Let G' be a generator of $\mathcal{A}$. Since $\mathcal{A}$ is not thin there is X and a pair of distinct morphisms $u, v:$ $: G \longrightarrow X$. Since G' is a generator there is a $\varphi : G' \longrightarrow G$ such that $u\varphi \neq v\varphi$. Similarly, there is a $\mu : G \longrightarrow G'$ such that $u\varphi\mu \neq v\varphi\mu$. Since $\mathcal{A}(G,G)=\{1_G\}$, $\varphi\mu = 1_G$, i.e. G is a retract of G'.

Remark. Less trival examples of absolute generators are e.g. the free algebras with one generator in some primitive classes of algebras (not in all, though). E.g., in the primitive classes of groups, abelian groups, semigroups, monoids, rings with or without units, lattices.

Proposition 2. Let G be an absolute generator of $\mathcal{A}$. Then μ is a g-monomorphism if and only if it is a G-injection.

Proof. It suffices to show that μ is a G-injection if and only if it is a G'-injection for any generator G' of $\mathcal{A}$. Yet, it suffices to show only one of these implications.

Assume that $\mu : G \longrightarrow X$ is a G-injection and G' is a generator of $\mathcal{A}$. Since G is an absolute generator, there are morphisms $\theta : G \longrightarrow G'$, $\theta' : G' \longrightarrow G$ with

----

x) A category $\mathcal{A}$ is said to be thin if for every pair A, $B \in \mathcal{A}$ there is at most one morphism $A \longrightarrow B$; so that $\mathcal{A}$ is not thin if and only if there are $A, B \in \mathcal{A}$ with distinct morphisms from A into B.

$\theta'\theta = id_G$. Let φ be a morphism : $B \longrightarrow X$, and for every morphism $G' \longrightarrow B$ there be a morphism $G' \longrightarrow A$ such that

$$G' \longrightarrow A \xrightarrow{\mu} X = G' \longrightarrow B \xrightarrow{\varphi} X \quad .$$

Now let $\beta : G \longrightarrow B$. Choosing $\beta' = \beta\theta' : G' \longrightarrow B$, by the above assumption, there is an $\alpha' : G' \longrightarrow A$ with $\mu\alpha' = \varphi\beta'$. Taking $\alpha = \alpha'\theta : G \longrightarrow A$, we have

$$\mu\alpha = \mu\alpha'\theta = \varphi\beta'\theta = \varphi\beta\theta'\theta = \varphi\beta \quad .$$

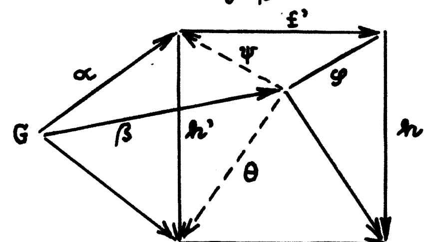
Since μ is a G-injection, there is a morphism $\psi : B \longrightarrow A$ such that $\varphi = \mu\psi$. Thus, μ is a G'-injection.

§ 2. Properties and relations to the other definitions.

Proposition 1. g-monomorphisms are preserved under compositions, intersections, pullbacks and under left divisions in the weak sense x).

All these statements are easily verified, we will prove only one e.g. for pullbacks.

Let f', h' be a pullback of f , h , where f is a g-monomorphism. Let φ be a morphism and let there be a generator G such that for every β there is an α with $f'\alpha = \varphi\beta$.



----

x) i.e. the dividing factor is required to be a monomorphism.

Then

$$(h\varphi)\beta = h(\varphi\beta) = hf'\alpha = fh'\alpha = f(h'\alpha) .$$

Since f is a g-monomorphism, there is a θ such that $f\theta = h\varphi$. Since the square is a pullback, there is a ψ such that $f'\psi = \varphi$. It finishes the proof.

Since g-monomorphisms coincide with general subspaces in the category of Hausdorff spaces (see § 5), the epimorphism assumption does not suffice to make a g-monomorphism an isomorphism. Instead of the epimorphisms it is necessary to put a stronger assumption.

<u>Definition</u>. A morphism $\varphi : X \longrightarrow Y$ is said to be g-<u>onto</u> if there is a generator G such that for every $\beta : G \longrightarrow Y$ there is an $\alpha : G \longrightarrow X$ such that $\beta = \varphi\alpha$.

It is easy to see that every g-onto morphism is an epimorphism.

<u>Proposition 2</u>. If μ is a g-monomorphism and g-onto, then μ is an isomorphism.

<u>Proof</u>. Let $\mu : A \longrightarrow B$ be a g-onto morphism, then there is a generator G such that for every $\beta : G \longrightarrow B$ there is an $\alpha : G \longrightarrow A$ with $\beta = \mu\alpha$, consequently $1 . \beta = \mu . \alpha$, and, since μ is a g-monomorphism, there is a $\psi : B \longrightarrow A$ such that $\mu\psi = 1$.

Thus, μ is a monomorphism and a retraction, so that it is an isomorphism.

About the relations of g-monomorphisms with the well-known definitions, at first we have the following hierarchy

that is shown in [6] :
equalizer $\Longrightarrow$ regular monomorphism $\Longrightarrow$ strong monomorphism $\Longrightarrow$ extremal monomorphism.

Proposition 3. In a category having coproducts, every strong monomorphism is a g-monomorphism. Consequently in a category having colimits, every extremal monomorphism is a g-monomorphism.

Proof. Let $\mu : A \longrightarrow X$ be a strong monomorphism, $\varphi : B \longrightarrow X$ a morphism and there be a generator G such that for every $\beta : G \longrightarrow B$ there is an $\alpha_\beta : G \longrightarrow A$ with $\mu\alpha_\beta = \varphi\beta$. Let ${}^{\langle G,B\rangle}G$ be a coproduct with the injections ν_β.

Then, for the family $\{\beta \mid \beta \in \langle G, B\rangle\}$, there is a $\lambda : {}^{\langle G,B\rangle}G \longrightarrow B$ such that $\lambda\nu_\beta = \beta$ for every β.

λ is an epimorphism because, if $f\lambda = g\lambda$, then $f\lambda\nu_\beta = g\lambda\nu_\beta$, i.e. $f\beta = g\beta$ for every β, consequently, since G is a generator, $f = g$.

Moreover, for the family $\{\alpha_\beta\}$ induced from β, there is a $\theta : {}^{\langle G,B\rangle}G \longrightarrow A$ such that $\theta\nu_\beta = \alpha_\beta$ for every β. It is clear that $\mu\theta = \varphi\lambda$ because

$$\mu\theta\nu_\beta = \mu\alpha_\beta = \varphi\beta = \varphi\lambda\nu_\beta \qquad \text{for every } \beta .$$

Since μ is a strong monomorphism, there is a $\psi : B \longrightarrow A$ such that $\mu\psi = \varphi$. So that μ is a g-monomorphism.

The consequence follows from the fact that, in a category with pushouts, strong monomorphism and extremal monomorphism coincide [x].

A similar proposition for regular monomorphisms may be proved without any assumption of the category in the question.

<u>Proposition 4</u>. Every regular monomorphism is a g-monomorphism.

<u>Proof</u>. Let $\mu : A \longrightarrow X$ be a regular monomorphism, φ be a morphism $: B \longrightarrow X$. Let G be a generator satisfying the condition in the definition of g-monomorphism.

Let $x\mu = y\mu$, we shall show that $x\varphi = y\varphi$. Suppose that $x\varphi \neq y\varphi$. Since G is a generator, there is a β such that $x\varphi\beta \neq y\varphi\beta$. Then we have an α such that $\mu\alpha = \varphi\beta$. Thus, $x\mu\alpha \neq y\mu\alpha$ which is a contradiction.

Now, since the definition of regular monomorphism, there is a ψ such that $\mu\psi = \varphi$, which shows that μ is a g-monomorphism.

§ 3. <u>Subobjects in concrete categories</u>.

A concrete category $(\mathcal{A}, \mathcal{U})$ is a category $\mathcal{A}$ together with a faithful functor $\mathcal{U}$ of $\mathcal{A}$ into the category of sets. The functor $\mathcal{U}$ is often referred to as its forgetful functor. On the other hand, considering $\mathcal{U}$ as a functor naturally equivalent with the functor $\mathcal{A}(G, -)$ for a generator G it looks to be useful to find the relation of

----

x) This is directly verified or dually follows from a similar statement formulated for strong and extremal epimorphisms that is found in [6].

$\mathcal{U}$ -subobjects defined below with g-monomorphisms. In some cases shown below (see the proposition 2, § 4), these two considerations of $\mathcal{U}$ are the same.

Definition 1. A monomorphism $\mu : A \longrightarrow B$ in a concrete category $(\mathcal{A}, \mathcal{U})$ is called a $\mathcal{U}$ -subobject if $\mathcal{U}(\mu)$ is one-to-one and if for every $f : \mathcal{U}(C) \longrightarrow \mathcal{U}(A)$ such that there is a $\varphi : C \longrightarrow B$ with $\mathcal{U}(\varphi) = \mathcal{U}(\mu) . f$, there is a $\psi : C \longrightarrow A$ such that $f = \mathcal{U}(\psi)$.

This concept is given in [4] in a form more general and under another name.

The monomorphism requirement of μ in the definition actually follows by the assumption that $\mathcal{U}$ is faithful and $\mathcal{U}(\mu)$ is one-to-one. Moreover, the faithfulness implies: $\varphi = \mu\psi$. It is also easy to show that $\mathcal{U}$ -subobjects are determined up to a naturally equivalence x).

In the relation to g-monomorphism we have

Proposition 1. Let $\mathcal{U} : \mathcal{A} \longrightarrow \mathcal{S}et$ be naturally equivalent to $\langle G, - \rangle$ for a generator G of $\mathcal{A}$. Then $\mathcal{U}$ -subobjects are exactly the G-injections. Consequently, if, moreover, G is an absolute generator, then the $\mathcal{U}$ -subobjects coincide with the g-monomorphisms.

The proposition follows immediately comparing the definitions together with these observations:

- The correspondence of $\beta : G \longrightarrow C$ to $\alpha : G \longrightarrow A$ in the definition of G-injections interchanges with a con-

----

x) That is: if $\mathcal{U}$ is naturally equivalent to $\mathcal{U}'$, then the $\mathcal{U}$ -subobjects coincide with the $\mathcal{U}'$ -subobjects.

struction of the function $f : \langle G, C\rangle \longrightarrow \langle G, A\rangle$ in the definition of $\mathfrak{U}$ -subobjects.

- The equality $\mu\alpha = \varphi\beta$ is equivalent to the equality $\mathfrak{U}(\mu).f = \mathfrak{U}(\varphi)$.

Let $(\mathcal{A}, \mathfrak{U})$ be a concrete category. A concrete category $(\mathcal{B}, \mathcal{V})$ is called a <u>full concrete subcategory</u> of $(\mathcal{A}, \mathfrak{U})$ if $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ and $\mathcal{V} = \mathfrak{U} | \mathcal{B}$. It is evident that every $\mathfrak{U}$ -subobject in $\mathcal{B}$ is a $\mathcal{V}$ -subobject. The converse is true only with some assumptions adding to the categories $\mathcal{A}$ and $\mathcal{B}$.

<u>Definition 2.</u> A concrete category $(\mathcal{A}, \mathfrak{U})$ is called a category with nice decompositions if every morphism of $\mathcal{A}$ may be written as $\varphi = \mu . \eta$ with μ a $\mathfrak{U}$ -subobject and $\mathfrak{U}(\eta)$ onto.

<u>Proposition 2.</u> Let $(\mathcal{A}, \mathfrak{U})$ be a category with nice decompositions. Let $(\mathcal{B}, \mathcal{V})$ be a full concrete subcategory of $(\mathcal{A}, \mathfrak{U})$ containing with each object all its $\mathfrak{U}$ - subobjects. Then μ is a $\mathcal{V}$ -subobject if and only if μ is a $\mathfrak{U}$ -subobject and $\mu \in \mathcal{B}$.

<u>Proof</u>. It suffices to prove that if $\mu : A \longrightarrow B$ is a $\mathcal{V}$ -subobject, it is a $\mathfrak{U}$ -subobject. We have $\mathfrak{U}(\mu) = \mathcal{V}(\mu)$ one-to-one.

Let $f : \mathfrak{U}(C) \longrightarrow \mathfrak{U}(A)$ be such that there is a $\varphi : C \longrightarrow B$ with $\mathfrak{U}(\varphi) = \mathfrak{U}(\mu).f$. Since $(\mathcal{A}, \mathfrak{U})$ has nice decompositions, we have $\varphi = C \xrightarrow{\eta} C' \xrightarrow{\nu} B$ with $\mathfrak{U}(\eta)$ onto and ν a $\mathfrak{U}$ -subobject. Consequently, C' is in $\mathcal{B}$.

On the other hand, we can write $f = m . e$ with m

one-to-one and e onto. Comparing the two decompositions of $\mathfrak{U}(\varphi)$:

$$\mathfrak{U}(\nu) . \mathfrak{U}(\eta) \quad \text{and} \quad (\mathfrak{U}(\mu) . m) . e \quad ,$$

we see that there is a one-to-one and onto mapping h : $: \mathfrak{U}(C') \longrightarrow X$ such that

$$\mathfrak{U}(\mu) . m . h = \mathfrak{U}(\nu) \text{ and } h . \mathfrak{U}(\eta) = e \quad .$$

The first of these equations may be rewritten as $V(\mu) . m . h = V(\nu)$, so that there is a $x : C' \longrightarrow A$ such that $V(x) = \mathfrak{U}(x) = m . h$. Put $\psi = x\eta$, we obtain $\mathfrak{U}(\psi) = \mathfrak{U}(x) . \mathfrak{U}(\eta) = m . h . \mathfrak{U}(\eta) = m . e = f$.

At last, let us notice that regular monomorphisms are stronger than $\mathfrak{U}$ -subobjects with only an assumption for $\mathfrak{U}$ more general than the one in the proposition 1.

<u>Proposition 3</u>. Let $(\mathcal{A}, \mathfrak{U})$ be a concrete category. Then every regular monomorphism μ with $\mathfrak{U}(\mu)$ one-to-one is a $\mathfrak{U}$ -subobject.

<u>Proof</u>. Let $\mu : A \longrightarrow B$ and $f : \mathfrak{U}(C) \longrightarrow \mathfrak{U}(A)$ be such that there is a $\varphi : C \longrightarrow B$ with $\mathfrak{U}(\mu) . f = \mathfrak{U}(\varphi)$.

The equality $\alpha\mu = \beta\mu$ implies the one $\alpha\varphi = \beta\varphi$. Really, we have $\mathfrak{U}(\alpha\varphi) = \mathfrak{U}(\alpha)\mathfrak{U}(\varphi) = \mathfrak{U}(\alpha\mu)f = \mathfrak{U}(\beta\mu)f = \mathfrak{U}(\beta\varphi)$, and the fact follows from the faithfulness of $\mathfrak{U}$.

Since μ is a regular monomorphism, there is a ψ with $\mu\psi = \varphi$. Then $\mathfrak{U}(\mu) . \mathfrak{U}(\psi) = \mathfrak{U}(\varphi) = \mathfrak{U}(\mu) . f$, so that, since $\mathfrak{U}(\mu)$ is one-to-one, $f = \mathfrak{U}(\psi)$.

§ 4. Categories $S((F_i)_{i \in I})$ and concrete form of subobjects.

The notion of $S((F_i)_{i \in I})$ categories was introduced in [3].

Definition 1. Let $(F_i)_{i \in I}$ be a system of set functors x) indexed by elements of a set I. The category $S((F_i)_{i \in I})$ is defined as follows:

the objects are couples $(X, (\kappa_i)_{i \in I})$ where X is a set and $(\kappa_i)_{i \in I}$ a system of sets $\kappa_i \subset F_i(X)$,

the morphisms from $(X, (\kappa_i)_{i \in I})$ into $(Y, (\rho_i)_{i \in I})$ are triples $((X, (\kappa_i)_{i \in I}), f, (Y, (\rho_i)_{i \in I}))$ such that f is a mapping $X \longrightarrow Y$ and for every i such that F_i covariant, $F_i(f)(\kappa_i) \subset \rho_i$, and for every i such that F_i contravariant, $F_i(f)(\rho_i) \subset \kappa_i$.

Unless otherwise stated, $S((F_i)_{i \in I})$ will be understood as a concrete category endowed by the forgetful functor $\mathfrak{U}$.

Definition 2. A morphism $((X, (\kappa_i)_i), f, (Y, (\rho_i)_i))$ in a $S((F_i)_i)$ category is said to be an injection if:

1) f is one-to-one.

2) For F_i covariant, $\kappa_i = F_i(f)^{-1}(\rho_i)$.

For F_i contravariant, $\kappa_i = F_i(f)(\rho_i)$.

The condition 2) means that if F_i is covariant (con-

----

x) This term is used to show covariant or contravariant functors from the category of sets and mappings into itself.

traveriant resp.) for $i \in I$, $\varkappa_i$ is the largest (smallest resp.) subset of $F_i(X)$ such that $((X,(\varkappa_i)_i), f, (Y,(\sigma_i)_i))$ is a morphism in $S((F_i)_i)$. The definition looks like a generalization of the notions usually understood as subobjects in current concrete categories.

Proposition 1. $\mu = ((A,(a_i)_i), m, (B,(b_i)_i))$ is an injection if and only if it is a $\mathfrak{U}$-subobject.

Proof. Let μ be an injection. Let $f : C \longrightarrow A$ be a mapping, $((C,(c_i)_i), g, (B,(b_i)_i))$ a morphism such that $g = m . f$. In order to prove that μ is a $\mathfrak{U}$-subobject, we have to show that $((C,(c_i)_i), f, (A,(a_i)_i))$ is a morphism.

Let F_i be covariant, we have

$$F_i(m)F_i(f)(c_i) = F_i(g)(c_i) \subset b_i$$

so that

$$F_i(f)(c_i) \subset F_i(m)^{-1}(b_i) = a_i \quad .$$

If F_i is contravariant, we have

$$F_i(f)(a_i) = F_i(f)F_i(m)(b_i) = F_i(g)(b_i) \subset c_i \quad .$$

On the other hand, let μ be a $\mathfrak{U}$-suobject. Consider the identity $id : A \longrightarrow A$ and the morphism $((A,(\bar{a}_i)_i), m, (B,(b_i)_i))$ where

$$\bar{a}_i = F_i(m)^{-1}(b_i) \qquad \text{or} \qquad F_i(m)(b_i)$$

according to the variance of F_i. Since μ is a $\mathfrak{U}$-subobject, we have $((A,(\bar{a}_i)_i), id, (A,(a_i)_i))$ a morphism. Hence, for covariant F_i, $\bar{a}_i \subset a_i$ and since μ

is a morphism, $a_i \subset \bar{a}_i$ so that $a_i = \bar{a}_i$. Analogously, for contravariant F_i , $a_i \subset \bar{a}_i$ and since μ is a morphism, $a_i \supset \bar{a}_i$, so that $a_i = \bar{a}_i$. Thus, μ is an injection.

<u>Proposition 2</u>. Let $\mathcal{A}$ be a full concrete subcategory of $S((F_i)_{i \in I})$ such that

1) whenever $\mu : A \longrightarrow B$ is an injection and $B \in |\mathcal{A}|$ then μ is in $\mathcal{A}$.

2) There is an object $G = (1, (a_i)_i)$ x) in $|\mathcal{A}|$ such that for all $(1, (\kappa_i)_i)$ in $\mathcal{A}$, $((1, (a_i)_i), id, (1, (\kappa_i)_i))$ is a morphism, i.e. $a_i \subset \kappa_i$ for covariant, $a_i \supset \kappa_i$ for contravariant F_i .

Then

a) $U | \mathcal{A}$ is naturally equivalent to $\langle G, - \rangle$.

b) g-monomorphisms in $\mathcal{A}$ are exactly the injections.

<u>Proof</u>. Let $\hat{X} = (X, (s_i)_i)$ be in $\mathcal{A}$, $x \in X$. There is exactly one morphism $\nu(\hat{X}, x) : G \longrightarrow \hat{X}$ sending 0 to x . (Really, there is obviously at most one. On the other hand, denote by j the mapping $1 \longrightarrow X$ sending 0 to x . We have, by 1), $(1, (\kappa_i)_i) \in \mathcal{A}$ where $\kappa_i = F_i(j)^{-1}(s_i)$ for F_i covariant and $\kappa_i = F_i(j)(s_i)$ for F_i contravariant. Hence, in the covariant case we have, by 2), $a_i \subset F_i(j)^{-1}(s_i)$, hence $F_i(j)(a_i) \subset s_i$; in the contravariant one directly $F_i(j)(s_i) \subset a_i$.)

Since evidently, for $\varphi : \hat{X} \longrightarrow \hat{Y}$ in $\mathcal{A}$,

----

x) 1 , as usual, designates a set of one point.

$$\varphi \cdot \nu(\hat{X}, x) = \nu(Y, \mathfrak{U}(\varphi)(x)) ,$$

we see easily that the system

$$\theta^{\hat{X}} : \mathfrak{U}(\hat{X}) \longrightarrow \langle G, \hat{X} \rangle$$

given by

$$\theta^{\hat{X}}(x) = \nu(\hat{X}, x)$$

forms a natural equivalence $\theta : \mathfrak{U} \longrightarrow \langle G, - \rangle$.

b) The object G in 2) is obviously an absolute generator. Hence, by the proposition 1, § 3 adding to a) and by the propositions 2, § 3 and 1, § 4, it suffices to show that $S((F_i)_i)$ is a category with nice decompositions.

Take a morphism $\varphi = ((X, (\kappa_i)_i), f, (Y, (s_i)_i))$. Decompose f into $X \xrightarrow{e} Z \xrightarrow{m} Y$ with m one-to-one and e onto. We see that

$$((Z,(t_i)_i), m, (Y,(s_i)_i)) \cdot ((X,(\kappa_i)_i), e, (Z,(t_i)_i))$$

with

$$t_i = F_i(m)^{-1}(s_i) \quad \text{for covariant } F_i ,$$

$$t_i = F_i(m)(s_i) \quad \text{for contravariant } F_i$$

is the required decomposition.

<u>Proposition 3.</u> In $S((F_i)_{i \in I})$ every $\mathfrak{U}$ -subobject is an equalizer.

<u>Proof</u>. Let $\mu = ((A,(a_i)_i), m, (B,(b_i)_i))$ be a $\mathfrak{U}$ -subobject, hence, by the proposition 1, an injection, so that

$$a_i = \begin{cases} F_i(m)^{-1}(b_i) & \text{for } F_i \text{ covariant,} \\ F_i(m)(b_i) & \text{for } F_i \text{ contravariant.} \end{cases}$$

Put $C = B \times 2/\sim$ where $2 = \{0,1\}$ and

$(b, j) \sim (b', j')$ iff $b = b'$ & $(j = j'$ or $\exists a \in A \mid b = m(a))$.

Define $f_0, f_1 : B \longrightarrow C$ by $f_j(b) = \overline{(b, j)}$ where the bar designates the equivalence class. For covariant F_i put $c_i = F_i(f_0)(b_i) \cup F_i(f_1)(b_i)$, and for contravariant F_i put $c_i = F_i(f_0)^{-1}(b_i) \cap F_i(f_1)^{-1}(b_i)$.

We obtain morphisms $\varphi_j = ((B, (b_i)_i), f_j, (C, (c_i)_i))$.

Obviously $\varphi_0 \mu = \varphi_1 \mu$.

Now, let for $\nu : ((X, (x_i)_i), g, (B, (b_i)_i))$ be $\varphi_0 \nu = \varphi_1 \nu$. Hence $f_0 g = f_1 g$, so that for $x \in X$, $(g(x), 0) \sim (g(x), 1)$, i.e. there is a $h(x) \in A$ with $g(x) = m(h(x))$. We obtain a mapping $h : X \longrightarrow A$ with $m \cdot h = g$. Now it suffices to show that $((X, (x_i)_i), h, (A, (a_i)_i))$ is a morphism. Let F_i be covariant, we have

$$F_i(m)(F_i(h)(x_i)) = F_i(g)(x_i) \subset b_i \ ,$$

hence

$$F_i(h)(x_i) \subset F_i(m)^{-1}(b_i) = a_i \ .$$

Let F_i be contravariant, we have

$$F_i(h)(a_i) = F_i(h) F_i(m)(b_i) = F_i(g)(b_i) \subset x_i \ .$$

The proposition is proved,

It is easy to see that $S((F_i)_{i \in I})$ is always cocomplete. Thus, by this proposition adding to the relations of the different types of monomorphisms which have been given in this paper, we have

Consequence. The notions of equalizer, regular monomorphism, strong and extremal monomorphism, g-monomorphism, $\mathfrak{U}$ -subobject and injection coincide in $S((F_i)_{i \in I})$.

§ 5. Applications.

Many current categories may be considered as a full concrete subcategory of a $S((F_i)_{i \in I})$ category. Such as: the category of topological spaces is a full concrete subcategory of $S(P^-)$ with P^- the preimage power set functor, the category of uniform spaces is of $S(P^- . Q_2)$ where Q_2 is the functor sending X into $X \times X$ and f into $f \times f$... . So that these following propositions are only the direct consequence of the proposition 2, § 4.

Proposition 1. Let $\mathcal{A}$ be a category of topological spaces which is closed with respect to subspaces. Then the g-monomorphisms are exactly the homeomorphisms onto subspaces.

Thus, the definition of g-monomorphism is suitable not only for the category of Hausdorff spaces, but also for the category of regular T_1 -spaces and so on.

A similar proposition is stated for the category of uniform spaces.

For further examples let us recall a category which is constructed in [3]. Let M be a set. M -ary relation on a set X is understood as a set $\varkappa \subset \langle M, X \rangle$. If $\varkappa$ is an M-ary relation on X and similarly ρ on Y, then $\varkappa\rho$-homomorphism of $(X, \varkappa)$ into (Y, ρ) is defined as a mapping $f : X \longrightarrow Y$ such that for every $\alpha \in \varkappa$, $f \cdot \alpha \in \rho$. We see easily that the category of sets with M-ary relations and their homomorphisms coincides with $S(Q_M)$ where $Q_M = \langle M, - \rangle$. Similarly as above we have

Proposition 2. Let M be a set, $\mathcal{A}$ a category of sets with M-ary relations and their homomorphisms. Let $\mathcal{A}$ be closed with respect to subsets with full subrelations. Then g-monomorphisms in $\mathcal{A}$ are exactly the isomorphisms onto subsets with full subrelations. In particular, in any category of graphs (sets with binary relations) closed with respect to full subgraphs, the g-monomorphisms are exactly the isomorphisms onto full subgraphs.

The consequence follows immediately by the observation that in an object $(1, \varkappa)$ here can be $|\varkappa|$ only 0 or 1.

R e f e r e n c e s

[1] ARDUINI P.: Monomorphisms and epimorphisms in abstract categories, Rend.Semin.Mat.Univ.Padova 42(1969), 135-166.

[2] DRBOHLAV K.: A note on epimorphisms in algebraic categories, Comment.Math.Univ.Carolinae 4(1963),81-85.

[3] HEDRLÍN Z. and PULTR A.: On categorical embeddings of topological structures into algebraic, Comment. Math.Univ.Carolinae 7(1966),377-400.

[4] HUŠEK M.: Categorical methods in topology, General Topology and its Relations to Modern Analysis and Algebra II,Proc.of the Second Prague Topological

Symposium,1966,Academia Prague,190-194.

[5] ISBELL J.R.: Subobjects,adequacy,completeness and categories of algebras, Rozpr.Mat.25(1964), 3-32.

[6] KELLY G.M.: Monomorphisms,epimorphisms and pullbacks, J.Austral.Math.Soc.9(1969),124-142.

[7] KUČERA L. and PULTR A: On a mechanism of choosing morphisms in concrete categories, General Topology and its Relations to Modern Analysis and Algebra III,Proc.of the Third Prague Topological Symposium,1971.Academia Prague, 263-265.

[8] SONNER J.: Canonical categories, Proc.of the Conf.on Cat.Alg.,La Jolla,1965.272-294.

[9] TRNKOVÁ V. and PULTR A.: Strong embeddings into categories of algebras, Ill.J.of Math.16(1972), 183-195.

Vien Toan
Uy ban khoa hoc ky thuat
39 Tran hung Dao
HA NOI
Viet-nam

Matematicko-fyzikální
fakulta
Karlova universita
Sokolovská 83, Praha 8
Československo

(Oblatum 12.10.1973)