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FURTHER NOTE ON FRÉCHET SPACES

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Abstract: This is a continuation of [1]. Further properties concerning C^*-embedding and complete separation of discrete closed countably infinite subsets of the Fréchet space Λ_∞ constructed by F.B. Jones are studied.

Key words: Fréchet space, Niemytzki space, C^*-embedding, complete separation.

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Answering a problem of J. Novák ([6, Problem 9]) it was shown in [1] that the space Λ_∞ [1]) constructed by F.B. Jones in [4] (as a Moore space which is not completely regular) is a sequentially regular Fréchet space which is not $\aleph_0$-completely regular [2]), i.e.

(A) There is a countable set $X \subset \Lambda_\infty$ and a point $x \in \Lambda_\infty - \overline{X}$ such that for every continuous function f on Λ_∞ we have $f(x) \in \overline{f[X]}$.

In the present paper (which is a continuation of [1]) it is shown that much more is true, viz. a discrete closed

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1) The space Λ_∞ was denoted by $(L_\infty, \lambda_\infty)$ in [1].

2) $\aleph_0$-regular was improperly used for $\aleph_0$-completely regular in [1]: henceforth only the latter will be used.

set $X \subset \Lambda_\infty$ satisfying (A) is constructed. This proves a conjecture of J. Novák. From the construction it follows that

(B) There is a discrete closed countable subspace Z which is not C^*-embedded (in the sense of [3]) in Λ_∞. Moreover, two propositions concerning complete separation of subsets of a discrete closed countable infinite set in Λ_∞ are given. Finally, it is proved that

(C) Λ_∞ is closed in every sequentially regular Fréchet space in which it is C^*-embedded.

The notation and results of [1] are used without explanation.

The following proposition is a slight modification of Proposition 1.2 in [5, p.444] :

Proposition 1. Let f be a continuous function on the Niemytzki space (L, λ). Then the function $h(x) = f((x, 0))$ is of the first Baire class.

Proof. For each $n \in N$, $h_n(x) = f((x, n^{-1}))$ is a continuous function of a real variable and $h_n \to h$.

In what follows E denotes the set of all rational points of D, i.e.

$$E = \{q \mid q = (x, 0),\ x \text{ rational}\}.$$

Proposition 2. Let f be a continuous function on the Niemytzki space (L, λ) such that $f[E] = 0$. Then for uncountably many points $q \in D$ we have $f(q) = 0$.

Proof. By Theorem 5.2 in [7] a function h is of the first Baire class if and only if $h^{\leftarrow}[V]$ is an F_σ -set for every open set $V \subset R$. Thus, in the above notation, $h^{\leftarrow}(0)$ is a G_δ-set. Since from the Baire category theorem it follows that a countable dense set of real numbers cannot be a G_δ -set, the set $h^{\leftarrow}(0)$ is uncountable and hence $f(q) = f((x,0)) = h(x) = 0$ for uncountably many $q \in D$.

Let X be the set of all rational points of the first "edge" of L_∞ ,i.e.

$$X = \{q_1 \mid q \in A \cap E\} \cup \{(q_1 ; q_2) \mid q \in B \cap E\} .$$

It follows that X is a discrete closed countable infinite subset of $(L_\infty , \lambda_\infty)$.

Proposition 3. Let f be a continuous function on $(L_\infty , \lambda_\infty)$ such that $f[X] = 0$. Then $f(p) = 0$.

Proof. Since (L_1 , λ_1) can be obviously regarded as a subspace of $(L_\infty , \lambda_\infty)$ it follows from Proposition 2 that $f(y) = 0$ for uncountably many $y \in Y$, where

$$Y = \{q_1 \mid q \in A\} \cup \{(q_1 ; q_2) \mid q \in B\} .$$

Now let ε be a positive real number and k a natural number. Since $f(y) = 0$ for uncountably many $y \in Y$ we have $f(x) = 0$ for uncountably many points of at least one of the sets $\{q_1 \mid q \in A\}$ and $\{(q_1 ; q_2) \mid q \in B\}$. Recall (cf.[4]) that if an open set $\mathcal{U} \subset L$ contains uncountably many points of one of the sets A , B , then $\lambda \mathcal{U}$ contains uncountably many points of the other. Using this result we obtain, after finitely many steps, $O_k(p) \cap f^{\leftarrow}[(-\varepsilon, \varepsilon)] \neq \emptyset$. Since

$\{O_k(p)\}$ is a fundamental system of neighbourhoods of p, we have $f(p) = 0$.

Let $Z = X \cup (p)$. Then Z is a discrete closed countable subset of L.

<u>Proposition 4</u>. The subspace $(Z, \lambda_{\infty/Z})$ is not C^*-embedded in $(L_\infty, \lambda_\infty)$.

<u>Proof</u>. Let f be a function defined on Z as follows:

$$f(x) = 0 \quad \text{for } x \in X,\ f(p) = 1 .$$

Then f is continuous on $(Z, \lambda_{\infty/Z})$ and it follows from Proposition 3 that f cannot be continuously extended onto $(L_\infty, \lambda_\infty)$.

<u>Proposition 5.</u> There is a discrete closed countable infinite set I in $(L_\infty, \lambda_\infty)$ and infinite subsets $I_1, I_2 \subset I$, $I_1 \cap I_2 = \emptyset$ which are not completely separated in $(L_\infty; \lambda_\infty)$.

<u>Proof</u>. In the same way as in Proposition 3 it can be proved that if

$$E' = \{q \mid q = (x + \sqrt{2}, 0),\ x \text{ rational}\}$$

then

$$X' = \{q_1 \mid q \in A \cap E'\} \cup \{(q_1; q_2) \mid q \in B \cap E'\}$$

is a discrete closed countable infinite subset in $(L_\infty, \lambda_\infty)$ such that X' and p are not completely separated. Put $I_1 = X$, $I_2 = X'$, $I = I_1 \cup I_2$ and the assertion is obviously satisfied.

Proposition 6. For every discrete closed countable infinite set I in $(L_\infty, \lambda_\infty)$ there are infinite subsets $I_1, I_2 \subset I$ which are completely separated.

Proof. Since I is a discrete closed countable infinite set in $(L_\infty, \lambda_\infty)$ it follows that $I-(p)$ is infinite and for some neighbourhood $O_{\varkappa}(p)$ of p we have $I-(p) \subset L_\infty - O_{\varkappa}(p)$. Consequently, there is an infinite subset $I_0 \subset I$ such that I_0 can be arranged into a one-to-one sequence $\langle z_i \rangle$ and either

a) Projection of every z_i lies in $L - D$,

or

b) For some fixed n every z_i is of the form $(q_n^{(i)}; q_{n+1}^{(i)})$ and if $q^{(i)} = (x_i, 0) \in D$ is the projection of z_i, then $\langle x_i \rangle$ is a strictly monotone, say increasing, sequence of real numbers.

In both cases, similarly as in [1, pp.414-415], a continuous function f on $(L_\infty, \lambda_\infty)$ can be constructed such that

$$f(z_{2i}) = 1 \quad \text{and} \quad f(z_{2i-1}) = 0, \quad i = 1, 2, 3, \ldots .$$

Proposition 7. Let $(L_\infty, \lambda_\infty)$ be a C^*-embedded subspace of a Fréchet space (S, σ). Then $\sigma L_\infty = L_\infty$.

Proof. Suppose that, on the contrary, $\sigma L_\infty - L_\infty \neq \emptyset$. Consequently, there is a one-to-one sequence $\langle x_n \rangle$ of points of L_∞ and a point $x \in S - L_\infty$, $x = \lim x_n$. Hence

$I = \bigcup (x_n)$ is a discrete closed countable infinite set in $(L_\infty, \lambda_\infty)$ and from Proposition 6 follows the existence of a continuous function f on $(L_\infty, \lambda_\infty)$ such that the sequence $\langle f(x_n) \rangle$ does not converge. Since f can be continuously extended over S we have a contradiction with $x = \lim x_n$.

Note. The reader familiar with [2] may have noticed that $(L_\infty, \lambda_\infty)$ has the property p. Further results concerning mutual relations between the property p and C^*-embedding of discrete closed countable subspaces of sequential (convergence) spaces are intended to be published elsewhere.

References

[1] R. FRIČ: A note on Fréchet spaces, Comment.Math.Univ. Carolinae 13(1972),411-418.

[2] R. FRIČ: Sequential envelope and subspaces of the Čech-Stone compactification, General Topology and its Relations to Modern Analysis and Algebra,III(Proc.Third Prague Topological Sympos., 1971).Academia,Prague,1972,123-126.

[3] L. GILLMAN and M. JERISON: Rings of continuous functions, Princeton,1960.

[4] F.B. JONES: Moore spaces and uniform spaces, Proc. Amer.Math.Soc.9(1958),483-486.

[5] S. MRÓWKA: Some comments on the author's example of a non-R-compact space, Bull.Acad.Polon.Sci.Sér. Sci.Math.Astronom.Phys.18(1970),443-448.

[6] J. NOVÁK: On some problems concerning the convergence spaces and groups, General Topology and its Relations to Modern Analysis and Algebra

(Proc.Kanpur Topological Conf.,1968).Academia, Prague,1971,219-229.

[6] R. SIKORSKI: Funkcje rzeczywiste.Tom I, Warszawa,1958.

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