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ON CHANGES OF INPUT/OUTPUT CODING I [x)]

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Abstract: Two classes of partial recursive functions corresponding to the intuitive notion of changes of input-output coding are introduced and two relations in the set of all enumerations of partial recursive functions are derived from them. Then tools of the theory of recursive functions are used to investigate the given structures.

Key words: Enumeration of partial recursive functions, acceptable enumeration.

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§ 1. Introduction. A large number of various computing devices has been designed for evaluating arithmetic functions. The evaluation of an arithmetic function by such a device is not direct – the device performs a mapping from a set of constructive objects (inputs) to a set of constructive objects (outputs) and it is necessary to interpret inputs and outputs as numbers. In other words, given a computing device, there is a freedom left in coding of numbers

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by inputs and outputs. The different codings are not only more or less convenient for mathematical or practical purposes, but they can essentially change the power of the given device. A well-known (cf.[1], exercise 2-5) example is coding of integers in Turing machines. If numbers are coded by all finite tape configurations, then there exist simple partial recursive functions which cannot be evaluated by Turing machines, but if only inputs in a "canonical" form are used for coding numbers, then every partial recursive function can be evaluated by a Turing machine.

This is a motivation for introducing and investigating the concepts of i-dependence and o-dependence of enumerations of arithmetic functions.

An enumeration of arithmetic functions $(\alpha_i)_{i=0}^{\infty}$ is said to i-depend on an enumeration $(\beta_i)_{i=0}^{\infty}$ iff $(\alpha_i)_{i=0}^{\infty}$ can be derived from $(\beta_i)_{i=0}^{\infty}$ by a change of input coding such that the following conditions hold:

(i) only inputs (possibly not all) which were used in the "old" coding are used in the "new" coding,

(ii) different numbers are coded by different inputs,

(iii) the change of coding can be done effectively.

Similarly $(\alpha_i)_{i=0}^{\infty}$ is o-dependent on $(\beta_i)_{i=0}^{\infty}$ iff $(\alpha_i)_{i=0}^{\infty}$ can be derived from $(\beta_i)_{i=0}^{\infty}$ by a change of output coding which satisfies the conditions:

(i) only outputs (possibly not all) which were used for the "old" coding are used for the "new" coding,

(ii) the change of coding can be done effectively.

In this paper we use tools of the theory of recursive functions to study these two relations on the system of all effective enumerations of partial recursive functions.

§ 2 contains definitions of basic notions and a summary of their elementary algebraic properties, § 3 and § 4 are devoted to the investigation of i-dependence and o-dependence, respectively.

§ 2. Basic notions.

We shall use the following notation throughout the paper: $\mathcal{P}_n$, ($\mathcal{R}_n$) denotes the sets of all n -argument partial recursive (all recursive) functions,

$\mathcal{N} = \{0, 1, 2, \dots\}$,

id is the identical function $\mathcal{N} \to \mathcal{N}$,

fg denotes composition of functions f , g ($fg(x) = f(g(x))$) ,

$\mathcal{D}f$, $\mathcal{R}f$ denote domain and range of f respectively,

$\varphi(x)\downarrow$ stands for $x \in \mathcal{D}\varphi$,

$\varphi(x)\uparrow$ stands for $x \notin \mathcal{D}\varphi$ and

$f(x) \simeq g(y)$ stands for $(f(x)\downarrow \equiv g(y)\downarrow)$ & & $(f(x)\downarrow \Longrightarrow f(x) = g(y))$.

Every effective enumeration $M_0, M_1, M_2, \dots$ of computing devices for evaluating (partial) arithmetic functions yields an enumeration $m_0, m_1, m_2, \dots$ of (partial) mappings

$$m_i : I \longrightarrow O \ ,$$

where I and O are some sets of inputs and outputs, respectively. The arithmetic functions come on scene as late as a coding of numbers by elements of I and O is chosen.

We turn our attention only to the case when $m_0, m_1, m_2, \dots$ is an effective enumeration of effective mappings and there are effective isomorphisms (i.e. 1-1 onto mappings)

$$is_1 : \mathcal{N} \longrightarrow I \qquad \text{and} \qquad is_2 : O \longrightarrow \mathcal{N} \ .$$

Then the enumeration $m_0, m_1, m_2, \dots$ passes to an enumeration $\alpha_0, \alpha_1, \alpha_2, \dots$, where

$$\alpha_i = is_2 \, m_i \, is_1 \ .$$

As is_1, is_2 are isomorphisms, the enumeration $\alpha_0, \alpha_1, \dots$ conversely determines the enumeration $m_0, m_1, \dots$ (cf. Fig. 2.1)

$$\begin{tikzcd}
I \arrow[r, "m_i"] \arrow[d, shift right] & O \arrow[d, shift left, "is_2"] \\
\mathcal{N} \arrow[u, shift right, "is_1"'] \arrow[r, "\alpha_i"] & \mathcal{N} \arrow[u, shift left]
\end{tikzcd}$$

Fig. 2.1.

We can therefore, without loss of generality, deal only with effective enumerations of partial recursive functions.

The following definitions formally introduce the notions of i-dependence and o-dependence described in § 1.

Definition 2.1. 1) We say that a function $h \in P_1$ is an i-convention iff h is 1-1 and total.

$\mathcal{I}$ will denote the set of all i-conventions.

2) We say that a function $f \in P_1$ is an o-convention iff f is onto $\mathcal{N}$.

$\mathcal{O}$ will denote the set of all o-conventions.

Lemma 2.1. 1) $\mathcal{I}$ forms a monoid wrt the operation of composition of functions and with id as the identity element.

(I.e. (i) $f \in \mathcal{I}$ & $g \in \mathcal{I} \Longrightarrow fg \in \mathcal{I}$,

(ii) $f \circ id = id \circ f = f$ for all $f \in \mathcal{I}$.)

2) $\mathcal{O}$ forms monoid wrt the same operation and identity element.

Proof: Immediate.

Definition 2.2. We say that $(\alpha_i)_{i=0}^{\infty}$ is an effective enumeration of partial recursive functions iff there is a $\sigma \in P_2$ such that $\sigma(i,x) \simeq \alpha_i(x)$ for all $i, x \in \mathcal{N}$.

Note. 1) By "enumeration" we shall mean "effective enumeration of partial recursive functions" throughout this paper,

2) "enumeration α" and "enumeration (α_i)" will be abbreviations for "enumeration $(\alpha_i)_{i=0}^{\infty}$".

Definition 2.3. Let φ, ψ be two enumerations. We define:

1) φ i-depends on ψ via f ($\varphi \leq^{i} \psi$ via f) iff $f \in \mathcal{J}$ and $\psi_i f = \varphi_i$ for all $i \in N$.

φ o-depends on ψ via h ($\varphi \leq^{\sigma} \psi$ via h) iff $h \in \mathcal{O}$ and $h\psi_i = \varphi_i$ for all $i \in N$.

2) φ i-depends on ψ ($\varphi \leq^{i} \psi$) iff there is an $f \in \mathcal{J}$ such that $\varphi \leq^{i} \psi$ via f.

φ o-depends on ψ ($\varphi \leq^{\sigma} \psi$) iff there is an $h \in \mathcal{O}$ such that $\varphi \leq^{\sigma} \psi$ via h.

3) φ is i-equivalent to ψ ($\varphi \equiv^{i} \psi$) iff $\varphi \leq^{i} \psi$ & $\psi \leq^{i} \varphi$.

φ is o-equivalent to ψ ($\varphi \equiv^{\sigma} \psi$) iff $\varphi \leq^{\sigma} \psi$ & $\psi \leq^{\sigma} \varphi$.

Lemma 2.2. 1) Both relations $\leq^{i}$ and $\leq^{\sigma}$ are reflexive and transitive.

2) The relations $\equiv^{i}$ and $\equiv^{\sigma}$ are equivalence relations.

Proof. 1) follows immediately from Lemma 2.1.

2) follows from 1).

We give an illustration of the meaning of i-dependence and o-dependence.

Recall the enumeration of mappings $m_0, m_1, m_2, \dots$ and the enumeration $\alpha_0, \alpha_1, \alpha_2, \dots$ from the beginning of this paragraph. Every effective input coding of numbers can be identified with a mapping

$$in : \mathcal{N} \longrightarrow I \qquad \text{(number } n \text{ is coded by } in(n) \text{).}$$

Analogously, output coding of numbers can be identified with an onto mapping

$$\sigma : O \longrightarrow \mathcal{N} \qquad (x \in O \text{ is interpreted as } \sigma(x)$$

if $\sigma(x)\downarrow$, x is without interpretation otherwise).

Isomorphisms is_1, is_2 give a correspondence between in and i-convention $f = (is_1)^{-1} \circ in$ and a correspondence between σ and o-convention $h = \sigma \circ is_2$. Let $\beta \leq^i \alpha$ via f and $\gamma \leq^\sigma \beta$ via h. Then

$$(\gamma_i) = (\sigma \circ m_i \circ in)$$

is an enumeration of functions evaluated by $M_0, M_1, \ldots$ with input and output codings in and σ respectively.

Definition 2.4. 1) $[\alpha]^i$ denotes the i-class (i.e. equivalence class wrt the equivalence $\equiv^i$) containing the enumeration α. $[\alpha]^\sigma$ denotes the o-class containing the enumeration α.

2) $$[\alpha]^i \leq^i [\beta]^i \text{ iff } \alpha \leq^i \beta,$$
$$[\alpha]^\sigma \leq^\sigma [\beta]^\sigma \text{ iff } \alpha \leq^\sigma \beta.$$

Evidently $\leq^i$ and $\leq^\sigma$ are partial orderings of i-classes and o-classes, respectively.

The following theorem will often be used in this paper.

Theorem 2.3. If $A \subseteq \mathcal{N}$ is infinite then (A is recursively enumerable (r.e.) set)$\Longleftrightarrow$ (there is an $f \in \mathcal{I}$ such that $Rf = A$)$\Longleftrightarrow$ (there is an $h \in \mathcal{O}$ such that $Dh = A$).

For the proof of the theorem see e.g. [1], Chap. 5.

§ 3. i-dependence.

In this paragraph we shall formulate and prove some properties of the structure given by i-dependence. Namely that:

1. There exists a maximal i-class.
2. Every i-class is formed by enumerations which differ only by "recursive permutations of inputs".
3. Some important families of i-classes form relative upper semilattices wrt i-dependence. This does not hold for the family of all i-classes.

Many results concerning i-dependence can be trivially obtained from the well-known theorems about program transformations and that is why the following definition will be useful:

Definition 3.1. Enumerations α, β will be called dual iff $(\forall i, m \in N)(\alpha_i(m) \simeq \beta_m(i))$.

The existence of a maximal i-class immediately follows from the existence of an acceptable enumeration of partial recursive functions. Therefore we recall the definition of acceptable enumeration.

Definition 3.2. An enumeration φ of all functions from P_1 is called acceptable enumeration (AE) if for every $\sigma \in P_2$ there exists a $g \in R_1$ such that $(\forall i, x \in N)(\sigma(i,x) \simeq \varphi_{g(i)}(x))$.

Fact 3.1. An AE exists.

For example the enumerations of P_1 given by the standard enumerations of Turing machines are AE. For more informations about acceptable enumerations see e.g. [1] .

We shall often use the following result which is a straightforward consequence of the basic properties of AE.

Theorem 3.2. Let φ be an AE. Then an enumeration ψ is AE iff there exist $g_1, g_2 \in R_1$ such that

$$(\forall i \in N)(\varphi_i = \psi_{g_1(i)} \ \& \ \psi_i = \varphi_{g_2(i)})$$

The theorem can be probably easily verified by the reader. If not, see [1] .

Definition 3.3. We say that an i-class $[\varphi]^i$ is maximal i-class iff $[\varphi]^i \geq^i [\psi]^i$ for the arbitrary enumeration ψ .

The fact 3.1 and the following theorem evidently imply the existence of a maximal i-class.

Theorem 3.3. Let φ be an enumeration. Then ($[\varphi]^i$ is maximal i-class) $\Longleftrightarrow$ (the enumeration dual to φ is AE).

Proof. $\Longleftarrow$: Let $\hat{\varphi}$ dual to φ be an AE and let ψ be an arbitrary enumeration. Let γ be the function such that $\gamma(i,x) \simeq \psi_x(i)$ for all $i,x \in N$. Then $\gamma \in P_2$ and there is a $g \in R_1$ such that $\hat{\varphi}_{g(i)}(x) \simeq \gamma(i,x) \simeq \psi_x(i)$ for every $i,x \in N$. By the technique of "padding" (cf. [1], § 7.2) a 1-1 recursive function

g' can be found such that $\hat{\varphi}_{g(i)} = \hat{\varphi}_{g'(i)}$ for all $i \in \in \mathcal{N}$. Apparently

$$(\forall i, x \in \mathcal{N}) [\varphi_i g'(x) \simeq \hat{\varphi}_{g'(x)}(i) \simeq \psi_i(x)]$$

and the "if" part of the theorem is established.

$\Longrightarrow$: Let $\varphi \geq^i \psi$ for all ψ. We choose ψ such that $\hat{\psi}$ dual to ψ is AE. It follows from the preceding part of the proof that $\psi \equiv^i \varphi$ and therefore $g_1, g_2 \in R_1$ exist for which $(\forall i \in \mathcal{N}) [\psi_i = \varphi_i g_1 \,\&\, \varphi_i = \psi_i g_2]$. This yields

$$(\forall i \in \mathcal{N}) [\hat{\varphi}_{g_1(i)} = \hat{\psi}_i \,\&\, \hat{\psi}_{g_2(i)} = \hat{\varphi}_i]$$

for $\hat{\varphi}$ dual to φ. Consequently, $\hat{\varphi}$ is AE by Theorem 3.2.

<u>Corollary 3.4</u>. The maximal i-class exists.

<u>Proof</u>: Immediate.

We introduce several auxiliary concepts, which will be useful in further investigation of i-dependence.

<u>Definition 3.4</u>. Let α be an enumeration. We define

1) $k =_\alpha \ell$ iff $\alpha_k = \alpha_\ell$ $(k, \ell \in \mathcal{N})$.

2) Let A, B be r.e. sets. Then

$A \subseteq_\alpha B$ iff there exists 1-1 partial recursive function δ such that (i) $D\delta \supseteq A$,

(ii) $(\forall i \in D\delta) [\delta(i) =_\alpha i]$,

(iii) $\delta(A) \subseteq B$.

3) Let A, B, C be r.e. sets. We say that C is an α-

supremum of A, B iff (i) $A \subseteq_\alpha C$ & $B \subseteq_\alpha C$,

(ii) for every r.e. set D

$$(A \subseteq_\alpha D \ \&\ B \subseteq_\alpha D) \Longrightarrow C \subseteq_\alpha D \ .$$

4) Let A, B be r.e. sets. Then

$A \approx_\alpha B$ iff there is a 1-1 function $\sigma \in P_1$ such that

$$D\sigma \supseteq A \ \&\ \sigma(A) = B \ \&\ (\forall i \in D\sigma)[\sigma(i) =_\alpha i] \ .$$

Note 3.5. 1) $=_\alpha$ and $\approx_\alpha$ are evidently equivalence relations on N and the class of r.e. sets respectively.

2) The relation $\subseteq_\alpha$ is reflexive and transitive.

Many important properties of the structure given by i-dependence can be derived from the following basic lemma.

Lemma 3.6. Let φ be an enumeration and $\hat{\varphi}$ its dual enumeration, let $\alpha \leq^i \varphi$ via h, $\beta \leq^i \varphi$ via f. Then the following two conditions are equivalent.

(1) $\alpha \leq^i \beta$,

(2) $Rh \subseteq_{\hat{\varphi}} Rf$.

Before proving the lemma, we recall a result of recursive function theory.

Theorem 3.7. For every $f \in P_1$ there exists a partial recursive function g such that $Dg = Rf \ \&\ Rg \subseteq Df \ \&\ \&\ fg(i) = i$ for every $i \in Rf$.

For the proof of Theorem 3.7 see [1] . Given an $f \in \in P_1$ we shall use the symbol f^{-1} only for a partial recursive function satisfying the conditions of Theorem 3.7.

Proof of Lemma. (1) $\Longrightarrow$ (2): Let $g \in \mathcal{J}$ exist such that $\varphi_i h = \varphi_i f g$ for all $i \in N$. This is equivalent to the assertion

$$(*) \qquad fg(i) \equiv_{q} h(i) \text{ for all } i \in N .$$

The partial recursive function $\delta = fgh^{-1}$ is apparently 1-1, $D\delta = Rh$ and $\delta(Rh) = R\delta \subseteq Rf$. From $(*)$ it can be easily deduced that $\delta(i) \equiv_{q} i$ for every $i \in D\delta$. Apparently $Rh \subseteq_{q} Rf$.

(2) $\Longrightarrow$ (1): Let $Rh \subseteq_{q} Rf$. That is, a partial recursive 1-1 function δ exists for which $D\delta \supseteq Rh$, $\delta(Rh) \subseteq Rf$ and $(\forall i \in D\delta)[\delta(i) \equiv_{q} i]$. Define the function $g = f^{-1}\delta h$. As $\delta(Rh) \subseteq Rf$, g is recursive and evidently g is 1-1. Thereby $g \in \mathcal{J}$ and $fg(i) = ff^{-1}\delta h(i) = \delta h(i) \equiv_{q} h(i)$ for every $i \in N$. This implies $\varphi_i h = \varphi_i f g$ for all $i \in N$ and the lemma is proved.

The following theorem gives an interesting characterization of i-classes.

<u>Theorem 3.8</u>. For every two enumerations α and β, $\alpha \equiv^{i} \beta$ iff there exists a recursive permutation (i.e. recursive, 1-1, onto N function) such that $\alpha_i p = \beta_i$ for all $i \in N$.

The "if" part of the theorem is immediate; the "only if" part can be obtained simply by dualization from the next theorem.

Theorem 3.9. Let φ and ψ be enumerations such that there exist recursive 1-1 functions g, f satisfying the following conditions

a) $\varphi_{g(i)} = \psi_i$,

b) $\psi_{f(i)} = \varphi_i$ for all $i \in N$.

Then there exists a recursive permutation p such that $\varphi_{p(i)} = \psi_i$ for all $i \in N$.

For proof of the theorem see [2] .

Corollary 3.10. For every r.e. sets A, B and for every enumeration ε

$$(A \subseteq_{\varepsilon} B \;\&\; B \subseteq_{\varepsilon} A) \Longleftrightarrow A \approx_{\varepsilon} B \quad .$$

Proof: 1) The "only if" part is immediate from the definition.

2. We prove the "if" part. Assume $A \subseteq_{\varepsilon} B \;\&\; B \subseteq_{\varepsilon} A$. If one of the sets A, B is finite, then the other is also finite and, as the reader can easily verify, the condition $A \approx_{\varepsilon} B$ evidently holds. If A and B are infinite, then there exist $f, g \in \mathcal{J}$ such that $\mathcal{R}f = A$ and $\mathcal{R}g = B$ by the theorem 2.3. Let $\hat{\varepsilon}$ denote the enumeration dual to ε . Then $(\hat{\varepsilon}_i g) \leq^i (\hat{\varepsilon}_i f)$ and $(\hat{\varepsilon}_i f) \leq^i (\hat{\varepsilon}_i g)$ (cf. Lemma 3.6). By the preceding theorem a recursive permutation h exists such that $\hat{\varepsilon}_i f h = \hat{\varepsilon}_i g$ for all $i \in N$. Consequently

(*) $fh(i) =_{\varepsilon} g(i)$ for all $i \in N$.

Define the partial function $\sigma = f h g^{-1}$. Apparently σ is partial recursive and 1-1, $D\sigma = Rg$ and $R\sigma = Rf$. Furthermore, for every $i \in D\sigma$ there is $\sigma(i) = f h g^{-1}(i)$ and by $(*)$ there is $f h g^{-1}(i) =_{\varepsilon} g g^{-1}(i) = i$. This implies $Rf \approx_{\varepsilon} Rg$ and the corollary is proved.

The next theorem will enable us to describe the structure given by i-dependence for some enumerations of special interest (e.g. enumerations of primitive recursive functions, acceptable enumerations etc.).

__Theorem 3.11__. Let $[\varphi]^{i}$ be a maximal i-class and $\hat{\varphi}$ the enumeration dual to φ, let $\varepsilon_1 \leq^{1} \varphi$ via h , $\varepsilon_2 \leq^{i} \varphi$ via f . Then the i-classes $[\varepsilon_1]^{i}, [\varepsilon_2]^{i}$ have supremum wrt $\leq^{i}$ if and only if the sets Rf and Rh have an $\hat{\varphi}$ -supremum.

__Proof__: The theorem can be proved by a straightforward application of Lemma 3.6.

I. First we prove the "only if" part. Let us assume that supremum of $[\varepsilon_1]^{i}$ and $[\varepsilon_2]^{i}$ exists. Let $\varepsilon_3 \in \in sup([\varepsilon_1]^{i}, [\varepsilon_2]^{i})$. Then there exists an $g \in \mathcal{J}$ such that $\varepsilon_3 = (\varphi_i\, g)$. By Lemma 3.6 there is $Rg \supseteq_{\hat{\varphi}} Rf , Rh$. For every r.e. set D such that $D \supseteq_{\hat{\varphi}} Rf , Rh$, there exists a $d \in \mathcal{J}$ for which $Rd = D$. Define ε_4 by $\varepsilon_4 = = (\varphi_i\, d)$. $\varepsilon_4 \geq^{i} \varepsilon_3$, as ε_3 is in $sup([\varepsilon_1]^{i}, [\varepsilon_2]^{i})$. Hence $D \supseteq_{\hat{\varphi}} Rg$ by Lemma 3.6 and Rg is evidently $\hat{\varphi}$ -supremum of Rf and Rh .

II. We now prove the "if" part of the theorem. Assume that A is a $\hat{\varphi}$ -supremum of Rf , Rh. A is then an infinite r.e. set and a $g \in \mathfrak{I}$ exists such that $Rg = A$ (by Theorem 2.3). Let us define $\psi = (\varphi_i\, g)$. Then $\psi \geq^i \varepsilon_1$, $\psi \geq^i \varepsilon_2$ by Lemma 3.6. For every enumeration χ there is an $e \in \mathfrak{I}$ such that $\chi \leq^i \varphi$ via e and if $\chi \geq^i \varepsilon_1$, $\chi \geq^i \varepsilon_2$ then $Re \geq_{\hat{\varphi}} Rf, Rh$ and therefore $Re \geq_{\hat{\varphi}} A$ by assumption. Hence $\chi \geq^i \psi$ by the same lemma and the theorem follows.

The reader may have noticed that the assumption concerning the enumeration φ was not used in the part I of the proof. The following corollary which is a stronger version of the "only if" part of the theorem, therefore holds.

Corollary 3.12. Let φ be an arbitrary enumeration, $\hat{\varphi}$ the dual enumeration, let $\varepsilon_1 \leq^i \varphi$ via h, $\varepsilon_2 \leq^i \varphi$ via f. Then
(supremum of $[\varepsilon_1]^i, [\varepsilon_2]^i$ exists) $\Longrightarrow$ ($\hat{\varphi}$-supremum of Rf, Rh exists).

Definition 3.5. 1) $\mathcal{E}$ will denote the class of all enumerations.

2) $\mathcal{E}^{1-1}$ will denote the class of all enumerations (α_i) for which $i_0 \in N$ exists such that α_{i_0} is recursive 1-1 function.

3) $\mathcal{A}\mathcal{E}$ will denote the class of all acceptable enumerations.

4) Let ε be an arbitrary enumeration. Then $\mathcal{L}_\varepsilon$ will denote the class of i-classes defined as follows:

$$[\varepsilon']^i \in \mathcal{L}_\varepsilon \Longleftrightarrow \varepsilon' \leq^i \varepsilon \ .$$

Corollary 3.13. Let $\varepsilon \in \mathcal{E}^{1-1}$. Then $\mathcal{L}_\varepsilon$ is an upper semilattice wrt $\leq^i$.

Proof: Let φ be an enumeration belonging to the maximal i-class and let f be an i-convention such that $\varepsilon \leq^i \varphi$ via f. Let $\hat{\varphi}$ be the enumeration dual to φ.

Let us assume that $\varepsilon_1 \leq^i \varepsilon$ via g,

$$\varepsilon_2 \leq^i \varepsilon \text{ via } h .$$

It follows by the preceding theorem that $[\varepsilon_1]^i, [\varepsilon_2]^i$ have supremum iff Rfg, Rfh have a $\hat{\varphi}$-supremum. Obviously, $Rfg = f(Rg)$ and $Rfh = f(Rh)$. To establish the corollary it evidently suffices to prove that for every r.e. sets A, B a $\hat{\varphi}$-supremum of $f(A), f(B)$ exists.

By assumption there is a recursive α_{i_0} in ε such that $\alpha_{i_0}(x) \neq \alpha_{i_0}(y)$ for every $x \neq y$. This implies $\varphi_{i_0} f(x) \neq \varphi_{i_0} f(y)$ for every $x \neq y$ and consequently $x =_{\hat{\varphi}} y \iff x = y$ for every $x, y \in Rf$. Therefore $f(A) \cup f(B)$ is a $\hat{\varphi}$-supremum of $f(A), f(B)$.

In the next theorem we show that the membership in $\mathcal{E}^{1-1}$ and $\mathcal{A}\mathcal{E}$ respectively is hereditary wrt $\leq^i$.

Theorem 3.14. Let α, β be enumerations such that $\alpha \geq^i \beta$. Then

1) $\alpha \in \mathcal{E}^{1-1} \implies \beta \in \mathcal{E}^{1-1}$,

2) $\alpha \in \mathcal{A}\mathcal{E} \implies \beta \in \mathcal{A}\mathcal{E}$.

Proof: 1) Let $\beta \leq^i \alpha$ via h. Let α_{i_0} be recursive 1-1 function, then $\beta_{i_0} = \alpha_{i_0} h$ is again recursive 1-1 function and $\beta \in \mathcal{E}^{1-1}$.

2) Let $\beta \leq^i \alpha$ via h and let α be acceptable. Then γ such that $(\forall i,x)[\gamma(i,x) \simeq \alpha_i h(x)]$ is a partial recursive function of two variables and so (cf. Definition 3.2) a $g_1 \in \mathbb{R}_1$ exists for which

$$(\forall i,x)[\alpha_{g_1(i)}(x) \simeq \gamma(i,x) \simeq \beta_i(x)] .$$

Conversely, h^{-1} is a partial recursive function and δ for which $(\forall i, x \in N)[\delta(i,x) \simeq \alpha_i h^{-1}(x)]$ is also a partial recursive function. Consequently, a recursive function g_2 exists such that

$$(\forall i, x \in N)[\alpha_{g_2(i)}(x) \simeq \delta(i,x) \simeq \alpha_i h^{-1}(x)] .$$

Since h is 1-1, $h^{-1} h(x) = x$ for all $x \in N$; hence the condition

$$(\forall i, x \in N)[\beta_{g_2(i)}(x) \simeq \alpha_{g_2(i)} h(x) \simeq \alpha_i h^{-1} h(x) \simeq \alpha_i(x)]$$

holds. We summarize:

$$\alpha_{g_1(i)} = \beta_i ,$$
$$\beta_{g_2(i)} = \alpha_i \qquad \text{for all } i \in N .$$

It follows that β is acceptable by Theorem 3.2.

Note. The more general relation of enumerations which arises if arbitrary partial recursive functions instead of i-conventions are used, could be studied.

Evidently, if $(\alpha_i) = (\beta_i f)$ and $\alpha, \beta \in \mathcal{E}^{1-1}$, then f must be i-convention. Therefore i-conventions are just the functions which transform members of $\mathcal{E}^{1-1}$ in members of $\mathcal{E}^{1-1}$ (and members of $\mathcal{A}\mathcal{E}$ in members of $\mathcal{A}\mathcal{E}$).

Theorem 3.14 gives immediately

$$\varepsilon_1 \equiv^i \varepsilon_2 \Longrightarrow [(\varepsilon_1 \in \mathcal{E}^{1-1} \Longleftrightarrow \varepsilon_2 \in \mathcal{E}^{1-1}) \,\&\, (\varepsilon_1 \in \mathcal{A}\mathcal{E} \Longleftrightarrow \varepsilon_2 \in \mathcal{A}\mathcal{E})]$$

for every enumerations $\varepsilon_1, \varepsilon_2$ (i.e. every i-class is either disjoint with $\mathcal{E}^{1-1}$ or is contained in $\mathcal{E}^{1-1}$; analogously for $\mathcal{A}\mathcal{E}$).

Now Corollary 3.13 can be strengthened as follows.

__Corollary 3.15__. 1) $\mathcal{E}^{1-1}/\equiv^i$ forms a relative upper semilattice wrt $\leq^i$ (i. every pair of i-classes from $\mathcal{E}^{1-1}/\equiv^i$ has a supremum in $\mathcal{E}^{1-1}/\equiv^i$ whenever it has an upper bound in $\mathcal{E}^{1-1}/\equiv^i$).

2) $\mathcal{A}\mathcal{E}/\equiv^i$ form a relative upper semilattice wrt $\leq^i$.

__Proof__: Immediate from Cowollary 3.13 and Theorem 3.14.

The corollary gives in a sense the strongest possible result - we show that "relative upper semilattice" cannot be replaced by "upper semilattice" in the previous corollary.

__Fact 3.16__. There are two acceptable enumerations φ, ψ which have no upper bound in $\mathcal{E}^{1-1}$

__Proof__: Choose an arbitrary $\alpha \in \mathcal{A}\mathcal{E}$. There is an $i_0 \in N$ such that $\alpha_{i_0}(x) = 0$ for all $x \in N$. Let us define

$$\varphi_i = \begin{cases} \alpha_{\frac{i}{2}} & \text{if } i \text{ is even,} \\ \alpha_{i_0} & \text{if } i \text{ is odd,} \end{cases}$$

$$\psi_i = \begin{cases} \alpha_{\frac{i-1}{2}} & \text{if } i \text{ is odd,} \\ \alpha_{i_0} & \text{if } i \text{ is even.} \end{cases}$$

The reader can easily verify (cf. Theorem 3.2) that φ and ψ are acceptable enumerations. Let γ be an upper bound of φ and ψ, i.e. $f, g \in \mathcal{J}$ exist such that

$$\varphi \leq^i \gamma \quad \text{via } f ,$$

$$\psi \leq^i \gamma \quad \text{via } g .$$

Assume that γ_{k_0} is a recursive function. We prove that it cannot be 1-1.

(i) if k_0 is odd then $\gamma_{k_0} f(x) = \varphi_{k_0}(x) = \varphi_{k_0}(y) = = \gamma_{k_0} f(y)$ and γ_{k_0} is not 1-1, as $f(x) \neq f(y)$ for $x \neq y$;

(ii) if k_0 is even then $\gamma_{k_0} g(x) = \gamma_{k_0} g(y)$ for all x, y and hence γ_{k_0} is not 1-1.

As k_0 was chosen arbitrary, $\gamma \notin \mathcal{E}^{1-1}$.

We recall that $\mathcal{A}\mathcal{E} \subset \mathcal{E}^{1-1} \subset \mathcal{E}$. We exhibited $\varepsilon_1, \varepsilon_2 \in \mathcal{A}\mathcal{E}$ such that there exists no upper bound of

$\varepsilon_1, \varepsilon_2$ in $\mathcal{E}^{1-1}$. Therefore $\mathcal{A}\mathcal{E}/_{\equiv^i}$ and $\mathcal{E}^{1-1}/_{\equiv^i}$ are not upper semilattices. Since there is an upper bound in $\mathcal{E}$ for every pair of enumerations, the question arises whether $\mathcal{E}/_{\equiv^i}$ is the upper semilattice wrt $\leq^i$. The answer is negative.

Theorem 3.17. For every acceptable enumeration α there is an acceptable enumeration β such that the supremum of $[\alpha]^i$ and $[\beta]^i$ does not exist.

Proof: Let α be an arbitrary enumeration. There is an $\kappa \in R_1$ such that $\alpha_{\kappa(i)} = id$ for every $i \in N$. Moreover, by the technique of "padding" we can construct an increasing recursive function of the desired property. Therefore $R\kappa$ can be assumed to be recursive.

The function f for which $(\forall x)[f(x) \simeq \alpha_x(x)]$ is a partial recursive function. Let P be an algorithm evaluating the function f. We define

$$m(i,x) \simeq \begin{cases} x & \text{if P does not complete evaluating of } f(x) \text{ in } i \text{ steps,} \\ \uparrow & \text{if P completes evaluating of } f(x) \text{ in } i \text{ steps.} \end{cases}$$

m is partial recursive function and there is a $g \in R_1$ such that $\alpha_{g(i)}(x) \simeq m(i,x)$ for all $i, x \in N$. It follows from the definition of m that $\bigcap_{i \in N} D\alpha_{g(i)} = \overline{Df}$. Since the range of the function κ is recursive and r is 1-1, the effective enumeration β can be defined as follows.

$$\beta_i = \begin{cases} \alpha_i & \text{if } i \notin Rr , \\ \alpha_{g(j)} & \text{if } i = r(j) . \end{cases}$$

The reader can easily verify that recursive functions s_1, s_2 exist such that $\alpha_i = \beta_{s_1(i)}$ and $\beta_i = \alpha_{s_2(i)}$ for all $i \in N$. Therefore β is acceptable by Theorem 3.2.

Let us note that the following equivalence holds.

$$(*)\ [(\forall i \in N)(\alpha_i(x) \simeq \beta_i(x))] \Longleftrightarrow x \in \bigcap_i D\alpha_{g(i)} .$$

Recall that for the enumerations $\hat{\alpha}$, $\hat{\beta}$ dual to α and β, respectively, there is $(i =_{\hat{\alpha}} j) \Longleftrightarrow (i =_{\hat{\beta}} j) \Longleftrightarrow (i = j)$. It is now easy to verify the following crucial equivalence.

$$(**)\ [(\forall i \in N)(\alpha_i(x) \simeq \beta_i(y)] \Longleftrightarrow$$

$$\Longleftrightarrow [x = y \ \&\ x \in \bigcap D\alpha_{g(i)}] .$$

We define γ :

$$\gamma_i(x) \simeq \begin{cases} \alpha_i\left(\frac{x}{2}\right) & \text{if } x \text{ is even,} \\ \beta_i\left(\frac{x-1}{2}\right) & \text{if } x \text{ is odd.} \end{cases}$$

Obviously $\gamma \geq^i \alpha$ via h_1

and $\gamma \geq^i \beta$ via h_2, where

$h_1(x) = 2x$ & $h_2(x) = 2x+1$ for all $x \in N$. Let $\hat{\gamma}$ be the enumeration dual to γ. If $[\alpha]^i, [\beta]^i$ have supremum then Rh_1, Rh_2 have a $\hat{\gamma}$-supremum by Corollary 3.12. We shall exhibit that no $\hat{\gamma}$-supremum of Rh_1, Rh_2 exists. This will complete the proof of the theorem.

Let $M = \{2x ; x \in \bigcap D\alpha_{g(i)}\}$. Then the definition of γ and $(**)$ imply

$$(***) \quad i < j \Longrightarrow [(i =_{\hat{\gamma}} j) \Longleftrightarrow (j = i+1 \;\&\; i \in M)] .$$

Let now A be an arbitrary r.e. set such that $A \geq_{\hat{\gamma}} Rh_1$ and $A \geq_{\hat{\gamma}} Rh_2$. We prove that A is not $\hat{\gamma}$-supremum of Rh_1, Rh_2. By Definition 3.4 there are 1-1 partial recursive functions τ, σ such that

$$A \geq_{\hat{\gamma}} Rh_1 \quad \text{via } \tau ,$$
$$A \geq_{\hat{\gamma}} Rh_2 \quad \text{via } \sigma .$$

Let us define $C = \{x ; x$ even and $(\exists y)(y$ odd and $\tau(x) \simeq \sigma(y))\}$ C is r.e. set and $C \subseteq M$ by $(***)$. Recall the definition of f. Evidently the set $Rf = \overline{\bigcap D\alpha_{g(i)}}$ is creative and therefore $\overline{M}$ is creative. This implies that an $i_0 \in M \setminus C$ exists.

We define: $B = A \setminus \{\tau(i_0)\}$.

Apparently $Rh_2 \leq_{\hat{\gamma}} B$ via σ and $Rh_1 \leq_{\hat{\gamma}} B$ via $\overline{\tau}$, where

$$\bar{\tau}(x) \simeq \begin{cases} \tau(x) & \text{if } x \neq i_0 , \\ \sigma(i_0+1) & \text{if } x = i_0 , \end{cases}$$

$B \leq_{\hat{\gamma}} A$ as $B \subseteq A$. But $A \leq_{\hat{\gamma}} B$ cannot hold as

(i) $\tau(i_0), \sigma(i_0+1) \in A \mathbin{\&} \tau(i_0) \neq \sigma(i_0+1) \mathbin{\&}$

$$\mathbin{\&} \tau(i_0) =_{\hat{\gamma}} \sigma(i_0+1) ,$$

(ii) $\sigma(i_0+1) \in B$ and for every $j_0 \in B$

$$\sigma(i_0+1) \neq j_0 \Longrightarrow \sigma(i_0+1) \neq_{\hat{\gamma}} j_0 .$$

Consequently A is not $\hat{\gamma}$ -supremum of $R\mathfrak{h}_1, R\mathfrak{h}_2$. As A was arbitrary, the theorem is proved.

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