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ON GENERAL CONCEPT OF BASIC SUBGROUPS.II

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Abstract: The purpose of this paper is to continue the investigation of basic subgroups begun in [1]. As an application, there is given the complete description of cotorsion abelian groups and a description of homogeneous separable groups in terms of subdirect sums. Further, there is given a description of all the countable torsion-free abelian groups in terms of interdirect sums of indecomposable groups and a complete description of countable homogeneous torsion-free groups of the type $\tau \in \Omega_{(0,\infty)}$ which have the nonzero indecomposable direct summands only the groups of rank 1.

Key words: Basic subgroups, direct summands, idempotents, cotorsion groups, separable groups, decompositions into indecomposable groups, superdecomposable groups, subdirect and interdirect sums, homogeneous groups, countable groups and accessible groups.

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0. Introduction. Essentially, this paper develops the theory of basic subgroups as it was introduced in [1]. Throughout the paper a group G always stands for an abelian group. Concerning the terminology and notation, we refer to [3], 282, and [1], 745-746. By $\mathfrak{G}$ and I_G we understand the set of all the direct summands of G and the set of all the idempotents of $End(G) = Hom(G,G)$, respectively. If $H \in \mathfrak{G}$, then $\overline{H} = \{p \in I_G ; p(G) = H\}$.

In particular, there is an equivalence relation $\sim$ on I_G, which is given by $p_1 \sim p_2 \Longleftrightarrow p_1(G) = p_2(G)$. By 9.5, [3], 47, $p_1 \sim p_2 \Longleftrightarrow \exists (f \in End(G)) \{ p_2 = p_1 + p_1 f(1-p_1) \}$.
We shall frequently use the following notation:

$\langle S \rangle^*$ - the pure closure of a set $S \subset G$,

$\mathfrak{M}_G = \{ p \in I_G ; p(G)$ is a nonzero, indecomposable subgroup$\}$,

$dom(f)$ - the domain of a homomorphism f,

$H_p^G(x), H^G(x), T^G(x)$ - the p-height, generalized height and the type of $x \in G$ in G (if it cannot lead to a confusion, we shall simply write $H_p(x)$, $H(x)$ and $T(x)$).

$\Omega_{(0,\infty)}$ - the set of all types with components only 0 or ∞.

If $f \in Hom(G, W)$, where G and W are torsion-free, then f is strongly regular if $\forall (w \in im(f)) \exists (g \in G) \{ H(g) = H^{im(f)}(w)$ and $f(g) = w \}$.

H is a quasi-superdecomposable subgroup of G if there is no nonzero indecomposable direct summand of G in H. By an order relation we mean the total order relation. For convenience, we are going to introduce the following definition and proposition from [1], 746-747.

<u>Definition 0.1</u>. We shall say that B is a basic subgroup of a group G if

(i) $B = \langle \{ G_\alpha ; \alpha \in \Lambda \} \rangle$, where $0 \neq G_\alpha$ is an indecomposable subgroup of G, for $\forall (\alpha \in \Lambda)$,

(ii) $\langle \{G_\alpha ; \alpha \in K\} \rangle = \coprod_{\alpha \in K} G_\alpha$ and $\coprod_{\alpha \in K} G_\alpha$ is a direct summand of G, for every finite $K \subset \Lambda$,

(iii) the family $\{G_\alpha ; \alpha \in \Lambda\}$ is maximal with respect to the conditions (i) and (ii).

The family $\{G_\alpha ; \alpha \in \Lambda\}$ is called the basic system of G corresponding to B.

Proposition 0.2. Let B be a basic subgroup of G. Then

(0.2) $G \simeq H \oplus W$, $B \subset W$ implies that H is a superdecomposable group.

By [1],747, every group contains a basic subgroup B and $B = \coprod_{\alpha \in \Lambda} G_\alpha$ is a pure subgroup of G. However, the properties of basic subgroups are not so coherent as it might be thought. For example, in the Specker group $Z^{\aleph_0}$, $Z^{(\aleph_0)}$ is not a basic subgroup and there exists a countable subgroup G of $Z^{\aleph_0}$ containing $Z^{(\aleph_0)}$ such that $Z^{(\aleph_0)}$ cannot be extended to a basic subgroup $B = G$, despite the fact that G is free.

Similar constructions as we present here, are considered in [5] with respect to separable groups.

1. On quasi-basic systems. The proofs of the following two propositions are straightforward and hence omitted.

Proposition 1.1. Let G be a group. Then the map

$$\begin{aligned} \varphi : G &\longrightarrow I_G/\sim \\ H &\longmapsto \overline{H} \end{aligned}$$ is a bijection.

Proposition 1.2. Let G be a group and $A, B, C \in$ $\in \mathbb{G}$. Then the following are equivalent:

(i) $A = B \oplus C$,

(ii) $\forall (p \in \bar{A})\, \exists!(q \in \bar{B})\, \exists!(r \in \bar{C})\, \{p = q + r \text{ and } rq = 0\}$,

(iii) $\exists (p \in \bar{A})\, \exists (q \in \bar{B})\, \exists (r \in \bar{C})\, \{p = q + r \text{ and } rq = 0\}$.

Proposition 1.3. Let G be a group and $q, r \in I_G$. Then the following are equivalent:

(i) $(q + r) \in I_G$,

(ii) $rq + qr = 0$,

(iii) $rq = qr$ and $2rq = 0$,

(iv) $(r + rq), (q + rq)$ and rq are pairwise orthogonal idempotents.

Moreover, $r + rq = 0$ iff $(q + r)$ and r are orthogonal idempotents. Furthermore, if G has no direct summands isomorphic to $Z(2)$, then $(q + r) \in I_G$ iff $qr = rq = 0$.

Proof. Obviously (i) $\Longleftrightarrow$ (ii) and (iii) $\Longrightarrow$ (iv).
(ii) $\Longrightarrow rq + qrq = qrq + qr = 0 \Longrightarrow$ (iii) ,
(iv) $\Longrightarrow (r + rq)rq = 2rq = 0$, $rq(r + rq) = rqr + rq = 0$, $(q + rq)rq = qrq + rq = 0$ and $(q + rq)(r + rq) =$ $= qr + qrq + rqr + rq = 0 \Longrightarrow$ (ii).
In view of (i) - (iv), the equivalence $r + rq = 0$ iff $(q + r)$ and r are orthogonal idempotents is trivial.

If G has no direct summand isomorphic to $Z(2)$ we can easily show that the condition (iii) implies $\kappa q = o$. q.e.d.

Remark 1.4. The last condition of the proposition 1.3 is necessary as it can be seen from the following example. Suppose $G = Z(2) \oplus B$, where $p : G \longrightarrow Z(2)$ is the corresponding projection. Then $p + p = o \in I_G$ and $p^2 = p \neq o$.

Proposition 1.5. Let G be a group, $p \in End(G)$ and $q', q \in I_G$. Then the following are equivalent:

(i) $pq = o$,

(ii) $q' \sim q \Longrightarrow pq' = o$.

Definition 1.6. We shall say that $\{p_\alpha \in I_G ; \alpha \in \Lambda\}$ is an orthogonal (quasi-orthogonal) system of a group G, if $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$ implies $p_\alpha p_\beta = o$ (if there is an order relation $\leq$ on Λ, such that $\alpha, \beta \in \Lambda$, $\alpha < \beta$ implies $p_\beta p_\alpha = o$). In the following, we shall denote it by OS and QOS, respectively.

Proposition 1.7. Every subset of I_G of any group G possesses a maximal OS and a maximal QOS with respect to the inclusion.

Proof. The existence of a maximal OS follows immediately by Zorn's Lemma. As to a maximal QOS, consider a subset $J \subset I_G$. Let $\mathfrak{N}$ be the family of all the QOS in J. Obviously $\mathfrak{N} \neq \emptyset$. Suppose that $\{S_\alpha ; \alpha \in \Lambda\} \subset \mathfrak{N}$ is a chain with respect to the inclusion and denote by $\leq_\alpha$ an order on S_α making S_α a quasi-orthogonal system of G. Define the reflexive and antisymmetric relation R on $S = \bigcup_{\alpha \in \Lambda} S_\alpha$ by $a, b \in S$, $aRb \Longleftrightarrow (a = b)$ or $(ba = 0$ and $ab \neq 0)$, and consider its transitive closure $\overline{R} = \bigcup_{n=1}^{\infty} R^n$, which is a partial order on S. For, it is sufficient to show the antisymmetricity. If $a\overline{R}b$ and $b\overline{R}a$, then $\exists(p_1, \dots, p_m, q_1, \dots, q_m \in S)$ such that $aRp_1, p_1Rp_2, \dots, p_mRb, bRq_1, \dots, q_mRa$. Now, there is $\beta \in \Lambda$ such that $a, b, p_1, \dots, p_n, q_1, \dots, q_m \in S_\beta$ and $a \leq_\beta p_1 \leq_\beta \dots \leq_\beta p_n \leq_\beta b \leq_\beta q_1 \leq_\beta \dots \leq_\beta q_m \leq_\beta a$. Hence $a = b$. Therefore, we can extend $\overline{R}$ into an order $\leq$ on S by Zorn's Lemma. If $a < b$, then $ba \neq 0$ implies $ab = 0$, hence bRa and consequently $b \leq a$, a contradiction. Therefore $S \in \mathcal{H}$, is an upper bound of $\{S_\alpha ; \alpha \in \Lambda\}$ and the Zorn's Lemma implies the existence of a maximal QOS in J, q.e.d.

<u>Proposition 1.8</u>. Let $S = \{R_p \in I_G ; \alpha \in \Lambda\}$ be a QOS of a group G. Then:

(i) For every finite $K \subset \Lambda$, there exists a QOS $S_K = \{p'_\alpha \in I_G ; \alpha \in \Lambda\}$ with $p'_\alpha \sim p_\alpha$, for $\forall(\alpha \in \Lambda)$, such that $\{p'_\alpha ; \alpha \in K\}$ is OS and $p'_\alpha p'_\beta = 0$, for $\forall(\alpha \in \Lambda, \beta \in K, \alpha \neq \beta)$. Further, $\bigcap_{\alpha \in K} \ker p_\alpha = \bigcap_{\alpha \in K} \ker p'_\alpha$ and $\bigcap_{\alpha \in \Lambda} \ker p_\alpha = \bigcap_{\alpha \in \Lambda} \ker p'_\alpha$.

(ii) If $S \in \mathfrak{M}_G$, then $B = \langle \{p_\alpha(G) ; \alpha \in \Lambda\} \rangle$ satisfies 0.1(i) and (ii). Moreover, if S is a maximal QOS in $\mathfrak{M}_G$, then B satisfies (0.2) and $\bigcap_{\alpha \in \Lambda} \ker p_\alpha$ is a quasi-superdecomposable subgroup of G .

(iii) If $\beta \in \Lambda$, $q_\alpha = (1 - p_\beta) p_\alpha$, for $\forall(\alpha \in \Lambda, \alpha \neq \beta)$, and $q_\beta = p_\beta$ then $S_\beta = \{q_\alpha \in I_G ; \alpha \in \Lambda\}$ is a QOS, where $\coprod_{\alpha \in \Lambda} p_\alpha(G) = \coprod_{\alpha \in \Lambda} q_\alpha(G)$, $p_\alpha(G) \cong q_\alpha(G)$, $\forall(\alpha \in \Lambda)$, and $\bigcap_{\alpha \in \Lambda} \ker p_\alpha = \bigcap_{\alpha \in \Lambda} \ker q_\alpha$.

<u>Proof</u>. (i) Let $K = \{\alpha_0, \dots, \alpha_m\} \subset \Lambda$, where $\alpha_0 < \alpha_1 < \dots < \alpha_m$ are in an order which makes S the QOS. Define $p'_\alpha = p_\alpha(1-p_{\alpha_0})\dots(1-p_{\alpha_m})$ for $\alpha < \alpha_0$; $p'_\alpha = p_\alpha(1-p_{\alpha_i})\dots(1-p_{\alpha_m})$ for $\alpha_{i-1} \leq \alpha < \alpha_i$, $i = 1, \dots, m$ and put $p'_\alpha = p_\alpha$ otherwise. S_K obviously possesses the desired properties.

(ii) Since $S \subset \mathfrak{M}_G$, the condition (i) implies that B satisfies 0.1(i) and (ii). If S is a maximal QOS in $\mathfrak{M}_G$ and $G = H \oplus W$, where $B \subset W$ and M is an indecomposable direct summand of H , then $H = M \oplus H'$ and for $\forall(\alpha \in \Lambda)$, $W = p_\alpha(G) \oplus W_\alpha$ i.e., $G = M \oplus H' \oplus p_\alpha(G) \oplus W_\alpha$. Suppose that $q : G \longrightarrow M$ and $\pi_\alpha : G \longrightarrow p_\alpha(G)$ are the corresponding projections with respect to the decompositions. Obviously $q\pi_\alpha = 0$, for $\forall(\alpha \in \Lambda)$. Since $\pi_\alpha \sim p_\alpha$, $q\pi_\alpha = 0$, by the proposition 1.5. Therefore the maximal condition on S yields $M = 0$. On the other hand, if $D \subset \bigcap_{\alpha \in \Lambda} \ker p_\alpha$ is an indecomposable direct summand of G and $q \in \overline{D}$ is arbitrary, we have $p_\alpha q = 0$, for $\forall(\alpha \in \Lambda)$. Hence, the maximal condition on S again yields $D = 0$.

(iii) S_β is obviously a QOS, for $\forall(\beta \in \Lambda)$. For the rest, it is sufficient to show that $p_\alpha(G) \oplus p_\beta(G) = q_\alpha(G) \oplus p_\beta(G)$, for $\forall(\alpha \in \Lambda, \alpha \neq \beta)$. By (i), $p_\alpha(G) \cap p_\beta(G) = q_\alpha(G) \cap p_\beta(G) = 0$, provided that $\alpha \neq \beta$. On the other hand, the equality $p_\beta(g_1) + p_\alpha(g_2) = p_\beta(g_1 + p_\alpha(g_2)) + (1 - p_\beta)p_\alpha(g_2)$ implies the desired result. q.e.d.

The assertion 1.8(ii) enables us to introduce the following definition.

<u>Definition 1.9</u>. We shall say that B is a quasi-basic subgroup of a group G if $B = \langle \{G_\alpha ; \alpha \in \Lambda\} \rangle$, where $\{G_\alpha ; \alpha \in \Lambda\} \subset \mathfrak{G}$, and there is a maximal QOS $\{p_\alpha \in \mathfrak{M}_G ; \alpha \in \Lambda\}$ in $\mathfrak{M}_G$ such that $p_\alpha \in \overline{G}_\alpha$, for $\forall(\alpha \in \Lambda)$. The family $\{G_\alpha ; \alpha \in \Lambda\}$ of subgroups of G will be called the quasi-basic system of G corresponding to B .

<u>Remark 1.10.</u> By 1.7 and 1.8, it follows that every group possesses a quasi-basic system and any quasi-basic system can be extended to a basic one.

<u>Theorem 1.11</u>. Let $\mathfrak{B} = \{G_\alpha ; \alpha \in \Lambda\}$ be a family of subgroups of a group G satisfying 0.1(i) and (ii), and suppose there is at most countable number of such α's that G_α is reduced, torsion-free. Then there exist $S_1 = \{p_\alpha \in \mathfrak{M}_G ; \alpha \in \Lambda\}$ and $S_2 = \{q_\alpha \in \mathfrak{M}_G ; \alpha \in \Lambda\}$ such that

(i) S_1 is QOS and $p_\alpha \in \overline{G}_\alpha$, $\forall(\alpha \in \Lambda)$,

(ii) S_2 is OS and $q_\alpha(G) \cong G_\alpha$, $\forall(\alpha \in \Lambda)$,

(iii) If G_α is either torsion or divisible then $q_\alpha = p_\alpha$,

(iv) $\coprod_{\alpha \in \Lambda} q_\alpha(G) = \coprod_{\alpha \in \Lambda} p_\alpha(G) = \coprod_{\alpha \in \Lambda} G_\alpha$,

(v) $\bigcap_{\alpha \in \Lambda} \ker p_\alpha = \bigcap_{\alpha \in \Lambda} \ker q_\alpha$.

Moreover, if $\mathcal{B}$ is a basic system, then $\{q_\alpha(G); \alpha \in \Lambda\}$ is again a basic system, corresponding to the basic subgroup $B = \coprod_{\alpha \in \Lambda} G_\alpha$, S_1 and S_2 are maximal QOS's in $\mathcal{M}_G$ and S_2 is a maximal OS in $\mathcal{M}_G$.

Proof. Write $\mathcal{B} = \{G_n; n \in \mathbb{N}\} \cup \{G_\alpha; \alpha \in \Lambda_1\} \cup \{G_\alpha; \alpha \in \Lambda_2\}$, where G_n is reduced, torsion-free, for $\forall (n \in \mathbb{N})$; G_α is divisible, for $\forall (\alpha \in \Lambda_1)$ and G_α is reduced, torsion, for $\forall (\alpha \in \Lambda_2)$. By 1.5 and 2.5, [1], 748 and 756, there is a disjunct decomposition $\Lambda_2 = \bigcup_{i=0}^{\infty} \Lambda_{2,i}$ such that

$$G = \coprod_{\alpha \in \Lambda_1} G_\alpha \oplus \coprod_{\alpha \in \Lambda_{2,0}} G_\alpha \oplus \ldots \oplus \coprod_{\alpha \in \Lambda_{2,n}} G_\alpha \oplus W_n,\ W_n = \coprod_{\alpha \in \Lambda_{2,n+1}} G_\alpha \oplus W_{n+1}$$

and $W_{n+1} \supset \coprod_{k \in \mathbb{N}} G_k$, for $\forall (n \in \mathbb{N})$. Hence we have an orthogonal system $S' = \{\kappa_\alpha \in \mathcal{M}_G; \alpha \in \Lambda_1 \cup \Lambda_2\}$, where $\kappa_\alpha \in \overline{G_\alpha}$, for $\forall (\alpha \in \Lambda_1 \cup \Lambda_2)$ and we can write

$$G = \coprod_{\alpha \in \Lambda_1} G_\alpha \oplus \coprod_{\alpha \in \Lambda_{2,0}} G_\alpha \oplus \ldots \oplus \coprod_{\alpha \in \Lambda_{2,n}} G_\alpha \oplus \coprod_{i=0}^{n} G_i \oplus W'_n, \text{ for } \forall (n \in \mathbb{N}).$$

Put $p_{i,n} \in \overline{G_i}$, for the corresponding projections of this decomposition, for $i = 0, \ldots, n$. If we define $p_n = p_{n,n}$, for $\forall (n \in \mathbb{N})$, we get the desired system $S_1 = S' \cup \{p_n; n \in \mathbb{N}\}$ (use the proposition 1.5). Now, define

$S_2 = S' \cup \{q_n ; n \in \mathbb{N}\}$, where $q_n = (1-p_0)\dots(1-p_{n-1})p_n$, for $\forall (n \in \mathbb{N})$. Similarly as in the proposition 1.8(iii) we can show $\coprod_{i=0}^{n} p_i(G) = \coprod_{i=0}^{n} q_i(G)$, $q_n(G) \cong G_n$, for $\forall (n \in \mathbb{N})$ and consequently S_2 is an OS in $\mathcal{M}_G$. If $\mathcal{B}$ is a basic system then $\{q_\alpha(G) ; \alpha \in \Lambda\}$ is obviously a basic system corresponding to the basic subgroup $B = \coprod_{\alpha \in \Lambda} G_\alpha$. According to 1.8(ii), S_1 and S_2 are maximal QOS's in $\mathcal{M}_G$ and consequently S_2 is a maximal OS in $\mathcal{M}_G$. The case, when the direct sum of all the G_α's which are reduced, torsion-free is a direct summand of G , can be treated by the same way. q.e.d.

Corollary 1.12. Every countable basic system is a quasi-basic one.

Proposition 1.13. Let $\mathcal{B} = \{G_\alpha ; \alpha \in \Lambda\}$ be a basic system of a group G such that either $\Lambda' = \{\alpha \in \Lambda ;\ G_\alpha$ is not alg. compact$\}$ is countable or $\coprod_{\alpha \in \Lambda'} G_\alpha$ is a direct summand of G . Then $\mathcal{B}$ is a quasi-basic system.

Proof. In both cases we can obviously construct a quasi-orthogonal system $S_1 = \{\kappa_\alpha \in \mathcal{M}_G ; \alpha \in \Lambda'\}$, such that $\kappa_\alpha \in \overline{G}_\alpha$, for $\forall(\alpha \in \Lambda')$ (if $|\Lambda'| \le \aleph_0$, use S_1 from the theorem 1.11). Suppose that S is a maximal QOS in $\bigcup_{\alpha \in \Lambda} \overline{G}_\alpha$ containing S_1 (the existence follows from 1.7). Since S can contain at most one element from each $\overline{G}_\alpha$, $\alpha \in \Lambda$, we have $S = \{p_\alpha \in \mathcal{M}_G ; \alpha \in \Gamma \subset \Lambda\}$, where $p_\alpha \in \overline{G}_\alpha$, for $\forall(\alpha \in \Gamma)$. Suppose that $\beta \in \Lambda \setminus \Gamma$ and write $B' = \coprod_{\substack{\alpha \in \Lambda \\ \alpha \neq \beta}} G_\alpha$. Since $B = \coprod_{\alpha \in \Lambda} G_\alpha$ is pure in G ,

$G_\beta \cong B/B'$ is pure in G/B' and since G_β is alg. compact ($\Lambda' \subset \Gamma$) we have $G/B' = (B/B') \oplus (G'/B')$ and consequently $G = G_\beta \oplus G'$, where $B' \subset G'$. Let q: $: G \longrightarrow G_\beta$ and $\pi_\alpha : G \longrightarrow G_\alpha$ be the corresponding projections with respect to the decompositions $G = G_\beta \oplus$ $\oplus G_\alpha \oplus G'_\alpha$, for $\forall(\alpha \in \Gamma)$. Since $\pi_\alpha \sim p_\alpha$ and $q\pi_\alpha =$ $= 0$, for $\forall(\alpha \in \Gamma)$, we have $q p_\alpha = 0$, for $\forall(\alpha \in \Gamma)$ by 1.5. Therefore it contradicts the maximality of S in $\bigcup_{\alpha \in \Lambda} \overline{G}_\alpha$ and consequently $\Gamma = \Lambda$. On the other hand, S is a maximal QOS in $\mathfrak{M}_G$ since any extension of S in $\mathfrak{M}_G$ would contradict the maximality of $\mathcal{B}$ by the proposition 1.8(ii). q.e.d.

<u>Corollary 1.14</u>. Let G be a group having the indecomposable direct summands only the alg. compact groups. Then $B \subset G$ is a basic subgroup iff B is a quasi-basic subgroup.

<u>Proof</u>. With respect to 1.13 it is sufficient to prove that every quasi-basic subgroup is a basic one, but it immediately follows by 1.6 [1], 750 and the proposition 1.8 (ii). q.e.d.

<u>Theorem 1.15</u>. Let $\{G_\alpha ; \alpha \in \Lambda\}$ be a quasi-basic system of a group G. Then there is a quasi-superdecomposable subgroup H of G such that for every finite $K \subset \Lambda$, G/H is isomorphic to a subdirect sum W of $\{G_\alpha ; \alpha \in \Lambda\}$ and $\coprod_{\alpha \in K} G_\alpha \subset W$.

<u>Proof</u>. Suppose that $S = \{p_\alpha \in \mathfrak{M}_G ; \alpha \in \Lambda\}$ is a maximal QOS in $\mathfrak{M}_G$ such that $p_\alpha \in \overline{G}_\alpha$, for $\forall(\alpha \in \Lambda)$.

Then $H = \bigcap_{\alpha \in \Lambda} \ker p_\alpha$ is a quasi-superdecomposable subgroup of G by 1.8(ii). If $K \subset \Lambda$ is finite, define $S_K = \{p'_\alpha ; \alpha \in \Lambda\}$ as in 1.8(i). For the rest it is sufficient to consider the homomorphism $\varphi : G \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha$ given by $g \longmapsto (p'_\alpha(g))_{\alpha \in \Lambda}$, since $\varphi / \coprod_{\alpha \in K} G_\alpha$ is the identity homomorphism and $\ker \varphi = H$, by 1.8(i). q.e.d.

Corollary 1.16. Let G be a group. Then there is a basic system $\{G_\alpha ; \alpha \in \Lambda\}$ of G and a quasi-superdecomposable subgroup H of G such that G/H is isomorphic to a subdirect sum of $\{G_\alpha ; \alpha \in \Lambda\}$.

Corollary 1.17. Let G be a homogeneous separable group. Then for every quasi-basic system $\{G_\alpha ; \alpha \in \Lambda\}$ of G and for every finite $K \subset \Lambda$, there exists a monomorphism $\varphi : G \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha$ such that $\varphi(G)$ is a subdirect sum of $\{G_\alpha ; \alpha \in \Lambda\}$ and $\varphi / \coprod_{\alpha \in K} G_\alpha$ is the identity homomorphism. In particular, G_α are pairwise isomorphic groups of rank 1. Moreover, if $|\Lambda| = \aleph_0$ then φ can be chosen in such a way that $\varphi(G)$ is an interdirect sum and $\varphi / \coprod_{\alpha \in \Lambda} G_\alpha$ is the identity.

Proof. According to 1.15, it is sufficient to show that $H = 0$. For, G/H being torsion-free implies that H is a pure subgroup of G and consequently $x \in H$ yields $\langle x \rangle^* \subset H$. Now, since H is a quasi-superdecomposable subgroup of G, $x = 0$ by 49.4 [2], 178, and similarly G_α's must be pairwise isomorphic groups of rank 1. If

$|\Lambda| = \aleph_0$, then the proofs of 1.11(ii),(iv) and (v) imply the desired result, q.e.d.

Corollary 1.18. Every separable homogeneous group is isomorphic to a subdirect sum of a system $\{G_\alpha ; \alpha \in \Lambda\}$,where G_α are pairwise isomorphic torsion-free groups of rank 1 .

Corollary 1.19. Every reduced, cotorsion and torsion-free group is isomorphic to a subdirect sum of (possibly nonisomorphic) groups of p -adic integers.

Proof. With regard to 1.15 it is sufficient to show that $H = 0$. Since G/H is torsion free, reduced, H is pure alg. compact and hence by 40.4 [3], 169, $H = 0$. q.e.d.

In view of [1] we can improve the result and since every reduced cotorsion group is direct sum of an adjusted and torsion-free, cotorsion group, the following two theorems give the complete description of cotorsion groups.

Theorem 1.20. The group G is reduced torsion-free and cotorsion iff there exists a family $\{G_\alpha ; \alpha \in \Lambda\}$ of groups of p -adic integers such that G is isomorphic to a minimal direct summand E of $\prod_{\alpha \in \Lambda} G_\alpha$ containing $\coprod_{\alpha \in \Lambda} G_\alpha$ and $E/\coprod_{\alpha \in \Lambda} G_\alpha$ is divisible, torsion-free.

Proof. Obviously, it is sufficient to prove only the necessary condition. Let G be a reduced torsion-free and cotorsion group and $\mathcal{B} = \{G_\alpha ; \alpha \in \Lambda\}$ be a basic system of G . By § 41 [3], the pure-injective hull E of $\coprod_{\alpha \in \Lambda} G_\alpha$ in $\prod_{\alpha \in \Lambda} G_\alpha$ is a minimal direct summand of $\prod_{\alpha \in \Lambda} G_\alpha$ containing $\coprod_{\alpha\in\Lambda} G_\alpha$ and $E/\coprod_{\alpha\in\Lambda} G_\alpha$ is torsion-free and divisible. On the other hand, $E \cong (\widehat{\coprod_{\alpha\in\Lambda} G_\alpha})$ and 1.12 [1], 753 implies the desired result. q.e.d.

Theorem 1.21. Let G be a reduced cotorsion group. Then G is adjusted iff there exists a family $\{G_\alpha;\ \alpha\in\Lambda\}$ of cyclic groups of prime power orders such that G/G^1 is isomorphic to the least direct summand E of $\prod_{\substack{p\in\mathbb{K}_B\\ n\in\mathbb{N}^+}} B_{p,n}$ containing $\coprod_{\alpha\in\Lambda} G_\alpha$, where $B_{p,n} = \coprod_{\alpha\in\Lambda_{p,n}} G_\alpha$, $\Lambda_{p,n} = \{\alpha\in\Lambda;\ G_\alpha \cong Z(p^n)\}$ and $\mathbb{K}_B = \{p\in\mathbb{P};\ (\coprod_{\alpha\in\Lambda} G_\alpha)_p \neq 0\}$, and $E/\coprod_{\alpha\in\Lambda} G_\alpha$ is divisible.

Proof. It is easy to see that by 2.9 [1], 760, the least direct summand of $\prod B_{p,n}$ containing $\coprod_{\alpha\in\Lambda} G_\alpha$ is the adjusted part of $\prod B_{p,n}$. If G has a torsion-free direct summand F, then since G^1 is fully invariant, $G^1 \cap F = F^1 = 0$ and it would contradict the hypothesis that G/G^1 has no nonzero torsion-free direct summand. Hence G is adjusted. Conversely, if $\mathfrak{B} = \{G_\alpha;\ \alpha\in\Lambda\}$ is a basic system of G and G is adjusted then G/G^1 is isomorphic to a direct summand E of $\prod B_{p,n}$ containing $\coprod_{\alpha\in\Lambda} G_\alpha$ by 2.7 [1], 760. Moreover, by [1], 751 and 756, $G/(G^1 \oplus \coprod_{\alpha\in\Lambda} G_\alpha) \cong E/\coprod_{\alpha\in\Lambda} G_\alpha$ is divisible. Hence E is an adjusted subgroup of $\prod B_{p,n}$. For, E is obviously reduced and cotorsion and if $E = F \oplus W$, where F is torsion-free and reduced, then $\coprod_{\alpha\in\Lambda} G_\alpha \subset E_t \subset W$ and $(\coprod_{\alpha\in\Lambda} G_\alpha) \cap F = 0$. Since

$E/\coprod_{\alpha\in\Lambda} G_\alpha \cong F \oplus (W/\coprod_{\alpha\in\Lambda} G_\alpha)$ is divisible, $F = 0$. By 55.5 [3], 238, $\prod B_{p,n} = A \oplus C$, where C is uniquely determined adjusted part of $\prod B_{p,n}$ such that $(\prod B_{p,n})_t \subset \subset C$, $C/(\prod B_{p,n})_t$ is divisible and A is torsion-free, cotorsion. Therefore C is a fully invariant subgroup of $\prod B_{p,n}$ and a minimal direct summand of $\prod B_{p,n}$ containing $\coprod_{\alpha\in\Lambda} G_\alpha$ by [1], 760. In fact, the uniqueness of the adjusted part implies that C is the least such a direct summand (it can also be seen from the following text). Now, if $\prod B_{p,n} = E \oplus W$, then $C = (C \cap E) \oplus (C \cap W)$ and since $\coprod_{\alpha\in\Lambda} G_\alpha \subset C \cap E$ and C is a minimal direct summand containing $\coprod_{\alpha\in\Lambda} G_\alpha$, $C \subset E$ and $E = C \oplus (A \cap E)$. On the other hand, E being adjusted implies $A \cap E = 0$ and consequently $E = C$. q.e.d.

2. The accessibility of groups.

Definition 2.1. We shall say that G is an accessible group if there exists a basic system $\{G_\alpha ; \alpha \in \Lambda\}$ of G and a homomorphism $f : G \longrightarrow \prod_{\alpha\in\Lambda} G_\alpha$ (called the accessible homomorphism) such that

(i) $\ker f$ is a quasi-superdecomposable subgroup of G,

(ii) $f(B) = \coprod_{\alpha\in\Lambda} G_\alpha$,

(iii) $\ker f \cap B = 0$,

where $B = \langle \{G_\alpha ; \alpha \in \Lambda\} \rangle$.

Theorem 2.2. Every group which possesses a basic system containing at most countable number of reduced torsion-free groups is accessible. Moreover, there is an accessible homomorphism for every such a basic system.

Proof. By 1.11 there is a basic system $\{G_\alpha ; \alpha \in \Lambda\}$ and an orthogonal system $S = \{q_\alpha \in \mathfrak{M}_G ; \alpha \in \Lambda\}$ such that $q_\alpha \in \overline{G_\alpha}$ and S is a maximal QOS in $\mathfrak{M}_G$. Hence the map

$$f : G \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha$$
$$g \longmapsto (q_\alpha(g))_{\alpha \in \Lambda}$$

is the desired accessible homomorphism by 1.8(ii). q.e.d.

Proposition 2.3. Let G be a group. Then for every basic system $\{G_\alpha ; \alpha \in \Lambda\}$ of G and for every automorphism ψ of $B = \langle \{G_\alpha ; \alpha \in \Lambda\} \rangle$ there exist disjoint subgroups A and K of G and a homomorphism $\varphi : A \oplus K \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha$ such that

(i) $B \subset A$,

(ii) $\varphi / B = \psi$,

(iii) $\ker \varphi = K$,

(iv) $G/(A \oplus K)$ is torsion,

(v) if G/A is not torsion, then $\prod_{\alpha \in \Lambda} G_\alpha / \varphi(A)$ is torsion,

(vi) if $|G| = |\Lambda| = \aleph_0$ and G is torsion-free, then $K = 0$.

Proof. Let $\mathfrak{N}$ be the set of all the monomorphisms f into $\prod_{\alpha \in \Lambda} G_\alpha$ such that $B \subset dom(f) \subset G$ and $f / B = \psi$.

Define $A = dom(g)$, where g is a maximal element of $\mathcal{H}$ by Zorn's Lemma and by K denote an A-high subgroup of G. Now, put

$$\varphi : A \oplus K \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha$$
$$(a, k) \longmapsto g(a)$$

Obviously, it is sufficient to prove only (v) and (vi). For, it both G/A and $\prod_{\alpha \in \Lambda} G_\alpha / \varphi(A)$ are not torsion, then the homomorphism g is not a maximal element of $\mathcal{H}$ contrary to our hypothesis. The conditions of (vi) imply that $\prod_{\alpha \in \Lambda} G_\alpha / \varphi(A)$ is not torsion (otherwise it would yield a contradiction with the cardinality of $\prod_{\alpha \in \Lambda} G_\alpha$), therefore by (v), G/A is torsion and consequently $K = 0$. q.e.d.

Theorem 2.4. Let $\{G_\alpha ; \alpha \in \Lambda\}$ be a basic system of a countable torsion-free group G. Then there exist subgroups H and A of G such that

(i) H is a quasi-superdecomposable subgroup of G,

(ii) G/H is isomorphic to an interdirect sum of $\{G_\alpha ; \alpha \in \Lambda\}$,

(iii) $B = \langle \{G_\alpha ; \alpha \in \Lambda\} \rangle \subset A$ and A is isomorphic to an interdirect sum of $\{G_\alpha ; \alpha \in \Lambda\}$,

(iv) G/A is either superdecomposable or torsion.

Proof. If B is a direct summand of G, define $A = B$ and for H put any direct complement of A which is superdecomposable by 0.2. Hence we can assume that B is not a direct summand of G. Put $H = \ker f$, where f is the accessible homomorphism corresponding to $\{G_\alpha ; \alpha \in \Lambda\}$ by

2.2 and construct A as it was done in 2.3. q.e.d.

Corollary 2.5. Let G be a countable torsion-free group. Then either G is a direct sum of a superdecomposable subgroup and indecomposable subgroups of G or G is the pure closure (in G) of an interdirect sum of a basic system of G and there is a quasi-superdecomposable subgroup H of G such that G/H is isomorphic to an interdirect sum of the basic system.

Lemma 2.6. Let G, K be torsion-free groups, $\varphi : G \longrightarrow K$ an epimorphism and $a \in K$. Then the following are equivalent:

(i) $\exists (x \in \varphi^{-1}(a)) \{H(x) = H(a)\}$,

(ii) $\forall (b \in \langle a \rangle^*) \exists (y \in \varphi^{-1}(b)) \{H(y) = H(b)\}$,

(iii) $\forall (b \in \langle a \rangle^*) \exists (y \in \varphi^{-1}(b)) \{T(y) = T(b)\}$,

(iv) $a = nb,\ n \in Z \Longrightarrow \exists (y \in \varphi^{-1}(b)) \{T(y) = T(b)\}$.

Proof. (i) $\Longrightarrow$ (ii). Let $b \in \langle a \rangle^*$, i.e. there are m, $n \in Z$ such that $mb = na$. By (i), there is $x \in \varphi^{-1}(a)$ such that $H(x) = H(a)$. Hence there exists $y \in G$ such that $nx = my$. For, m divides na and since $H(na) = H(nx)$, m must divide nx as well. Now, $m\varphi(y) = na = mb$ and consequently $\varphi(y) = b$, and $H(my) = H(nx) = H(na) = H(mb)$ implies $H(y) = H(b)$.

(ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) is obvious.

(iv) $\Longrightarrow$ (i). By (iv) we can assume that there is $y \in \varphi^{-1}(a)$ such that $T(y) = T(a)$ and since $H(y) \leq H(a)$, there is

$m = p_1^{k_1} \dots p_k^{k_k}$ such that $H(my) = H(a)$. Put $\bar{m} = p_1^{\ell_1} \dots p_k^{\ell_k}$, where $\ell_i = H_{p_i}(a) < \infty$, for $i = 1, \dots, k$. ($H_{p_i}(a) < \infty$, $i = 1, \dots, k$, since otherwise this particular p_i would be missing in the prime decomposition of m, a contradiction). Then there is $b \in K$ such that $\bar{m} b = a$ and by (iv) there is $z \in \varphi^{-1}(b)$ and $t \in \mathbb{N}^+$ such that $H(tz) = H(b)$. Since $H_{p_i}(b) = 0$, for $i = 1, \dots, k$, $(t, m) = 1$ and there are $u, v \in \mathbb{Z}$ such that $tu + mv = 1$. Put $x = tu\bar{m}z + mvy$. Then $\varphi(x) = (tu + mv)a = a$ and $H(a) = H(\bar{m}b) = H(\bar{m}tz) \leq$ $\leq H(\bar{m}tuz)$ and $H(a) = H(my) \leq H(mvy)$. Hence $H(a) \leq$ $\leq H(\bar{m}tuz) \cap H(mvy) \leq H(x)$. The converse $H(x) \leq H(a)$ is trivial. q.e.d.

Corollary 2.7. Every accessible homomorphism of a torsion-free, homogeneous group G is strongly regular.

Proof. Let $\{G_\alpha ; \alpha \in \Lambda\}$ be a basic system corresponding to a basic subgroup B of G and $\varphi : G \longrightarrow \prod_{\alpha \in \Lambda} G_\alpha = W$ be an accessible homomorphism. Consider an arbitrary $0 \neq x = (x_\alpha)_{\alpha \in \Lambda} \in \varphi(G)$ and an $y \in \varphi^{-1}(x)$. Obviously $T(y) \leq$ $\leq T^{\varphi(G)}(x)$ and there is $\alpha \in \Lambda$, such that $x_\alpha \neq 0$. Denote by $\bar{x}_\alpha = (\dots, 0, \dots, x_\alpha, \dots, 0, \dots) \in \coprod_{\alpha \in \Lambda} G_\alpha$. Since $\bar{x}_\alpha \in \varphi(G)$, there is $b_\alpha \in \varphi^{-1}(\bar{x}_\alpha) \cap B$ and $H(b_\alpha) = H^W(\bar{x}_\alpha) \geq$ $\geq H^W(x) \geq H^{\varphi(G)}(x)$. Since G is homogeneous, $T(y) = T(b_\alpha) \geq$ $\geq T^{\varphi(G)}(x)$. q.e.d.

Theorem 2.8. Let G be a separable, homogeneous group and H be a countable homogeneous subgroup of G of the same type τ as G. Then H is completely decomposable.

Proof. Let S be a pure subgroup of H of the finite rank n. According to [2], 174, it is sufficient to prove that H/S is homogeneous of the type τ. Denote by S^* the pure closure of S in G, which is again of the rank n. Obviously $S \subset H \cap S^*$. Conversely, if $h \in H \cap S^*$, then there is $m \in Z$ and $s \in S$ such that $mh = s$ and since S is pure in H, $h \in S$, i.e. $S = H \cap S^*$. Since $(H+S^*)/S^* \cong H/S$, all we have to show is that $H + S^*$ is homogeneous of the type τ. For, by [2], 178, $G = S^* \oplus W$ and consequently $H + S^* = S^* \oplus (W \cap (H + S^*))$. Hence $(H+S^*)/S^* \cong$ $\cong W \cap (H + S^*)$ and if $H + S^*$ is homogeneous of the type τ, H/S is also homogeneous of the same type τ. Now, if $0 \neq x \in (H+S^*)$, $x = h + s$, then $\tau = T^G(x) \geq$ $\geq T^{H+S^*}(x) \geq T^H(h) \cap T^{S^*}(s) = \tau$, q.e.d.

Theorem 2.9. Let G be a countable homogeneous, torsion-free group of the type $\tau \in \Omega_{(0,\infty)}$ and suppose that $\{G_n ; n \in \mathbb{N}\}$ is a basic system of G such that $\varkappa(G_n) = 1$. Then G is isomorphic to a direct sum of a completely decomposable homogeneous group and a superdecomposable group.

Proof. By 2.2, G is accessible and there is an accessible homomorphism $f : G \longrightarrow \prod_{n \in \mathbb{N}} G_n$, which is strongly regular by 2.7. Since $\prod_{n \in \mathbb{N}} G_n$ is homogeneous, separable group ([4], 338), and $H = f(G)$ satisfies the conditions of 2.8, H is completely decomposable, i.e. we can write $H = \coprod_{n=1}^{\infty} H_n$, where $r(H_n) = 1$ and H_n are pairwise isomorphic groups of the same type as G . Since $\ker f$ is a pure subgroup and $G/\ker f \cong \coprod_{n=1}^{\infty} H_n$, $\ker f$ is a direct summand of G by [2], 164. q.e.d.

Corollary 2.10. Every countable, torsion-free and homogeneous group of the type $\tau \in \Omega_{(0,\infty)}$ having the non-zero indecomposable direct summands only the groups of rank 1 is a direct sum of a completely decomposable and a super-decomposable group.

References

[1] BEČVÁŘ J., JAMBOR P.: On general concept of basic subgroups, Comment.Math.Univ.Carolinae 13(1972), 745-761.

[2] FUCHS L.: Abelian groups, Budapest 1958.

[3] FUCHS L.: Infinite abelian groups I, Acad.Press 1970.

[4] KRÓL M.: Separable groups, I, Bull.Acad.Pol.des Sci. (5)9(1961),337-344.

[5] MYŠKIN V.I.: Odnorodnye separabel'nye abelevy gruppy bez kručenija, Mat.sbornik 64(1964),3-9.

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