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GENERATION OF COREFLECTIONS IN CATEGORIES

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Abstract: This paper is concerned with generating of reflections and coreflections in categories. In the first section I give the fundamental construction of the paper, the category $K - F$, where F is a monocoreflector in a given category $\mathcal{O}\!\mathcal{L}$ and K is any class of objects from $\mathcal{O}\!\mathcal{L}$, and derive some properties of the notion. In the second part I give an example in the category of topological spaces, and make some remarks about bireflective subcategories. The third section deals with applications in the category of uniform spaces and uniformly continuous mappings.

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1.

Suppose $\mathcal{O}\!\mathcal{L}$ is a category. We shall denote, as usual, by $|\mathcal{O}\!\mathcal{L}|$ the class of objects and by $\mathcal{O}\!\mathcal{L}^m$ the class of morphisms of the category $\mathcal{O}\!\mathcal{L}$. The symbol $f: a \longrightarrow b$ (or $a \xrightarrow{f} b$) will denote a morphism from the object a to b.

1.1. Definition. Let $\mathcal{O}\!\mathcal{L}$ be a category, F a coreflector from $\mathcal{O}\!\mathcal{L}$ onto a coreflective subcategory $\mathcal{L}$. For $a \in |\mathcal{O}\!\mathcal{L}|$, let $\mu^a: F(a) \longrightarrow a$ denote the corresponding coreflection. Further let $\mathcal{K}$ be a class of objects of $\mathfrak{A}$. We shall say that an object $a \in |\mathfrak{A}|$ has the property $\mathcal{K}$ - - F if for every $b \in \mathcal{K}$, $(f: a \to b) \in \mathfrak{A}^m$ there exists $(g: a \to F(b)) \in \mathfrak{A}^m$ so that $\mu^b g = f$.

Let $\mathcal{K}$ - F denote the full subcategory of $\mathfrak{A}$ generated by all objects with the property $\mathcal{K}$ - F .

Analogously the dual definition: Let F be a reflector from $\mathfrak{A}$ onto a reflective subcategory $\mathfrak{B}$, for $a \in |\mathfrak{A}|$, $\mu^a :$ $: a \to F(a)$ the corresponding reflection, $\mathcal{K} \subset |\mathfrak{A}|$. We say that $a \in |\mathfrak{A}|$ has the property $\mathcal{K} * F$, if for every $b \in$ $\in \mathcal{K}$, $(f: b \to a) \in \mathfrak{A}^m$ there exists $(g: F(b) \to a) \in$ $\in \mathfrak{A}^m$ so that $g \mu^b = f$. We denote by $\mathcal{K} * F$ the corresponding full subcategory of $\mathfrak{A}$.

1.2. <u>Proposition</u>. Let $\mathfrak{A}$ be a category, F a coreflector in the category $\mathfrak{A}$. Let $\mathcal{K}$, L be two classes of objects of $\mathfrak{A}$. Then the following is true:

(a) If $\mathcal{K} \subset L$, then $L - F \subset \mathcal{K} - F$.

(b) $(\mathcal{K} \cup L) - F = \mathcal{K} - F \cap L - F$.

The proof follows immediately from the definition.

1.3. <u>Theorem</u>. Let $\mathfrak{A}$ be a cocomplete locally and colocally small category, F a monocoreflector in $\mathfrak{A}$, $\mathcal{K}$ any class of objects from $\mathfrak{A}$. Then $\mathcal{K}$ - F is a monocoreflective subcategory of $\mathfrak{A}$.

<u>Proof</u>: I shall use the criterion of monocoreflectivity given in [6]. It follows from there that to prove the monocoreflectivity of $\mathcal{K}$ - F it suffices to prove the following:

a) $\mathcal{K}$ - F is closed under isomorphisms. (This is evident.)

b) All coproducts of objects from $\mathfrak{K} - \mathrm{F}$ are in $\mathfrak{K} - \mathrm{F}$.

c) Coequalisers of diagrams $a \overset{f}{\underset{g}{\rightrightarrows}} b$, where $b \in |\mathfrak{K} - \mathrm{F}|$, are in $\mathfrak{K} - \mathrm{F}$.

a) Let $\{b_\alpha\}_{\alpha \in J}$ be a collection of objects from $\mathfrak{K} - \mathrm{F}$. Let $\{b_\alpha \xrightarrow{m_\alpha} \Sigma\, b_\alpha\}_{\alpha \in J}$ be their coproduct in the category $\mathcal{O}\!\mathcal{L}$. Further let $a \in \mathfrak{K}$, $(f: \Sigma\, b_\alpha \longrightarrow a) \in \mathcal{O}\!\mathcal{L}^m$. For every $\alpha \in J$ we have the morphism $(f m_\alpha : b_\alpha \longrightarrow a) \in \mathcal{O}\!\mathcal{L}^m$; $b_\alpha \in |\mathfrak{K} - \mathrm{F}|$, so there exists $(g_\alpha : b_\alpha \longrightarrow \mathrm{F}(a))$ so that for every $\alpha \in J$ there is $\mu^a g_\alpha = f m_\alpha$. Further for every $\alpha \in J$ we have the morphism $g_\alpha : b_\alpha \longrightarrow \mathrm{F}(a)$; consequently there exists exactly one $(g: \Sigma\, b_\alpha \longrightarrow \mathrm{F}(a)) \in \mathcal{O}\!\mathcal{L}^m$ such that for every $\alpha \in J$ there is $g m_\alpha = g_\alpha$. This implies that for every $\alpha \in J$ there is $\mu^a g m_\alpha = \mu^a g_\alpha = f m_\alpha$, from which $\mu^a g = f$. Consequently $\Sigma\, b_\alpha \in |\mathfrak{K} - \mathrm{F}|$.

b) Suppose given the diagram $a \overset{f}{\underset{g}{\rightrightarrows}} b$ in the category $\mathcal{O}\!\mathcal{L}$, $b \in |\mathfrak{K} - \mathrm{F}|$. Let $p : b \longrightarrow c$ be the coequaliser of (f, g) in the category $\mathcal{O}\!\mathcal{L}$. We are to show that $c \in |\mathfrak{K} - \mathrm{F}|$. Let $d \in \mathfrak{K}$, $(\bar{h} : c \longrightarrow d) \in \mathcal{O}\!\mathcal{L}^m$. Then $(\bar{h} p : b \longrightarrow d) \in \mathcal{O}\!\mathcal{L}^m$; consequently there exists $(h : b \longrightarrow \mathrm{F}(d)) \in \mathcal{O}\!\mathcal{L}^m$ so that $\mu^d h = \bar{h} p$. Since $\mu^d h f = \mu^d h g$ and μ^d is a monomorphism, $hf = hg$. From the limit property of coequalisers there exists exactly one $(u : c \longrightarrow \mathrm{F}(d)) \in \mathcal{O}\!\mathcal{L}^m$ such that $up = h$. Then $\mu^d u p = \mu^d h = \bar{h} p$ and p is an epimorphism, so that $\mu^d u = \bar{h}$. This implies that $c \in |\mathfrak{K} - \mathrm{F}|$ and the theorem is proved.

Further in this paper let $\mathcal{A}$ denote a complete, locally and colocally small category, F the monocoreflector from $\mathcal{A}$ onto a subcategory $\mathcal{B}$. If $a \in |\mathcal{A}|$, we shall denote by $\mu^a : F(a) \longrightarrow a$ the corresponding monomorphism given by the functor F.

Let us denote by F_K the monocoreflector in $\mathcal{A}$ onto the subcategory $K - F$. Let F, G be two monocoreflectors in $\mathcal{A}$, and $\mathcal{B}, \mathcal{C}$ the corresponding monocoreflective subcategories of $\mathcal{A}$. We shall write $F < G$ iff $\mathcal{B} \subset \mathcal{C}$.

1.4. Proposition. (a) If $F < G$, then $K - F \subset K - G$.

(b) $K - F = K - F_K$.

(c) F_K is the largest monocoreflector (in the order "$<$") with the property (b).

1.5. Proposition. Let $\mathcal{C} = K - F$ (monocoreflective in $\mathcal{A}$). Let $K_{\mathcal{C}} = \cup\{L \subset |\mathcal{A}| \mid \mathcal{C} = L - F\}$. Then:

(a) $K_{\mathcal{C}} - F = \mathcal{C}$,

(b) $K_{\mathcal{C}}$ is the largest class of objects from $\mathcal{A}$ (in the order given by inclusion) which fulfils (a). (So whenever $L - F = \mathcal{C}$, then $L \subset K_{\mathcal{C}}$.)

The proofs of the propositions 1.4 and 1.5 are evident.

1.6. Notes: (1) By 1.5, for every subcategory $\mathcal{C} = K - F$ of the category $\mathcal{A}$, there exists the largest class $L \subset |\mathcal{A}|$ such that $\mathcal{C} = L - F$. We shall call it the F-maxigenerator of $\mathcal{C}$ in $\mathcal{A}$ and denote it $L = K_{F\,max}$ or only K_m, if there will be no ambiguities.

We shall call the class $K \subset |\mathcal{A}|$ an F-maxigenerator if there exists a subcategory $\mathcal{C}$ of $\mathcal{A}$ such that

$\mathcal{C} = \mathcal{K} - F$, $\mathcal{K} = \mathcal{K}_m$.

(2) If $\mathcal{K}, \mathcal{L} \subset |\mathfrak{A}|$, then:

(a) $\mathcal{K} \subset \mathcal{K}_m$,

(b) $\mathcal{K} \subset \mathcal{L}$ implies $\mathcal{K}_m \subset \mathcal{L}_m$,

(c) $\mathcal{K}_m \cup \mathcal{L}_m \subset (\mathcal{K} \cup \mathcal{L})_m$,

(d) $(\mathcal{K}_m)_m = \mathcal{K}_m$.

1.7. <u>Proposition</u>. $\mathcal{K}_{F\max} = \{ b \in |\mathfrak{A}| \mid \forall a \in |\mathcal{K} - F| \ \forall (a \xrightarrow{f} b) \in \mathfrak{A}m \ \exists (a \xrightarrow{g} F(b)) \in \mathfrak{A}^m$ such that $\mu^b g = f \}$.

The proof follows immediately from 1.1 and 1.5.

1.8. <u>Proposition</u>. Let $\mathcal{C}$ be a subcategory of the category $\mathfrak{A}$. Let

$\mathcal{K}_{\mathcal{C}} = \{ b \in |\mathfrak{A}| \mid \forall a \in |\mathcal{C}| \ \forall (f : a \longrightarrow b) \in \mathfrak{A}^m \ \exists (g : a \longrightarrow F(b)) \in \mathfrak{A}^m$

such that $\mu g = f \}$.

Then $\mathcal{K} - F$ is the least subcategory in $\mathfrak{A}$ of the type $\mathcal{K} - F$ containing $\mathcal{C}$.

To prove this proposition it suffices to notice that $\mathcal{K}_{\mathcal{C}} - F = \bigcap \{ \mathcal{K} - F \mid \mathcal{C} \subset \mathcal{K} - F \}$ and that the intersection of a family of subcategories of the type $\mathcal{K} - F$ is again of this type.

1.9. <u>Note</u>. Evidently such $\mathcal{K}_{\mathcal{C}}$ is an F-maxigenerator. The category $\mathcal{K}_{\mathcal{C}} - F$ from the foregoing proposition we shall call the F-hull of $\mathcal{C}$ in $\mathfrak{A}$ and denote $F\mathit{hull}(\mathcal{C})$.
It is easy to see the validity of the following two propositions:

1.10. Proposition. Let K be an F-maxigenerator, $K - F \subset L - F$. Then $L \subset K$.

1.11. Proposition. Let K, L be F-maxigenerators; then $(K \cap L) - F = F\mathit{hull}(K - F \cup L - F)$.

1.12. Theorem. Be $\mathcal{C}$ a monocoreflective subcategory of a category $\mathcal{A}$, $\mathcal{B} \subset \mathcal{C}$, G the corresponding monocoreflector. Then $K_{\mathcal{C}} = \{x \in |\mathcal{A}| \mid F(x) = G(x)\}$.
In a special case: $K_{F\max} = \{b \in |\mathcal{A}| \mid F_K(b) = F(b)\}$.

(We understand by the equality an isomorphism in the category $\mathcal{A}$.)

Proof: Let $x \in K_{\mathcal{C}}$. There is $G(x) \in |\mathcal{C}|$. Let $\eta^x : G(x) \longrightarrow x$ be the monomorphism corresponding to the coreflector G. There exists $(\xi^x : G(x) \longrightarrow F(x)) \in \mathcal{A}^m$ such that $\mu^x \xi^x = \eta^x$. Since $\mathcal{B} \subset \mathcal{C}$, $F(x) \in |\mathcal{C}|$. Then there exists (exactly one) $(\vartheta^x : F(x) \longrightarrow G(x)) \in \mathcal{A}^m$ so that $\eta^x \vartheta^x = \mu^x$. Consequently $\eta^x \vartheta^x \xi^x = \mu^x \xi^x = \eta^x$, $\mu^x \xi^x \vartheta^x = \eta^x \vartheta^x = \mu^x$. The morphisms μ^x, η^x are monomorphisms, so $F(x)$ is isomorphic to $G(x)$.
Conversely let $F(x) = G(x)$, $b \in |\mathcal{C}|$, $(f : b \longrightarrow x) \in \mathcal{A}^m$. There is $G(b) = b$, so $(G(f) : b \longrightarrow G(x)) \in \mathcal{A}^m$. But $G(x) = F(x)$, hence, $x \in K_{\mathcal{C}}$.
There is a question: Is every monocoreflective subcategory $\mathcal{C}$ in $\mathcal{A}$ containing $\mathcal{B}$ of the type $K - F$ for some K? The answer is negative - see example 1.15.

The following two propositions are easy:

1.13. Proposition. $K_{F\max}$ is closed under retracts.

1.14. <u>Proposition</u>. $\mathcal{A} - F = \mathcal{B}$, $\mathcal{B} - F = \mathcal{A}$. $\mathcal{A}$, $\mathcal{B}$ are maxigenerators. If α is an ordinal number, we denote by P_α the set of all ordinal numbers less than α . The set P_α is well ordered, hence, we can consider P_α as a thin category, where for every β, $\gamma \in P_\alpha$ there is a morphism from β to γ if and only if $\beta \leq \gamma$.

1.15. <u>Example</u>: We consider the category P_{ω_1+1}. P_{ω_1+1} is cocomplete (because every subset of P_{ω_1+1} has a supremum lying in P_{ω_1+1}). P_{ω_1+1} is a small category so it is locally and colocally small. It is easy to verify that $\mathcal{B} \subset P_{\omega_1+1}$ is monocoreflective iff for every subset $B \subset \subset \mathcal{B}$ there is $\mathit{Sup}\, B \in \mathcal{B}$. Let $\mathcal{B}$ be monocoreflective in P_{ω_1+1} , F the corresponding monocoreflector. Let $K \subset P_{\omega_1+1}$, $t \in P_{\omega_1+1}$. Let $K_t = \mathit{Inf}\,\{\alpha \in K \mid \alpha \geq t\}$. There is $t \in K - F$ if and only if $k_t = \omega_1$ or there exists $v \in \mathcal{B}$ such that $t \leq v \leq k_t$.

Let us consider $\mathcal{B} = P_{\omega_1+1}$ now. P_{ω_0+2} is monocoreflective in P_{ω_1+1} , $P_{\omega_0+1} \subset P_{\omega_0+2}$. Suppose there exists $K \subset P_{\omega_1+1}$ such that $P_{\omega_0+2} = K - F$. It is easy to see that every element from K is less than $\omega_0 + 2$. Hence, $\omega_0 + 3 \in K - F = P_{\omega_0+2}$, which is a contradiction.

From the definition of a maxigenerator and from 1.11 we get easily:

1.16. <u>Proposition</u>. The intersection of two F -maxigenerators is again an F -maxigenerator.

1.17. <u>Definition</u>. Let $\mathcal{A}$ be a cocomplete, locally and colocally small category, F a monocoreflector in $\mathcal{A}$. Let us denote by A_F the class of all subcategories $\mathcal{C} \subset \mathcal{O}\!\mathcal{L}$ such that there exists $K \subset |\mathcal{O}\!\mathcal{L}|$ so that $\mathcal{C} = K - F$. On A_F we define the operations " $\wedge$ ", " $\vee$ ":

$$K - F \wedge L - F = (K \cup L) - F ,$$

$$K - F \vee L - F = (K_m \cap L_m) - F .$$

1.18. <u>Theorem.</u> $(A_F, \wedge, \vee)$ forms a complete distributive lattice with $0, 1$.

The proof follows immediately from 1.16 and 1.17. The role of 0 is played by the category $\mathcal{L}$ and the role of 1 is played by the category $\mathcal{O}\!\mathcal{L}$.

1.19. <u>Definition.</u> If $K \subset |\mathcal{O}\!\mathcal{L}|$, we define by induction:

$$K^1 - F = K - F ,$$

$$K^{n+1} - F = (K^n - F) - F \quad \text{for } n \geq 1 .$$

1.20. <u>Proposition.</u> (a) $K_m \cap K - F = \mathcal{L}$.

(b) If $n \geq 1$, then $K^{n+1} - F \cap K^n - F = K^{n+1} - F \cap (K^n - F)_{Fmax} = \mathcal{L}$.

The proof is evident.

1.21. Analogously to 1.2 - 1.20 we can formulate and prove the duals 1.2′ - 1.20′ if we begin with the definition of $K * F$.

<u>References:</u> The construction $K - F$ is given also in [11] and a special case in [8]. The construction in the category of uniform spaces is described in [3].

2.

We shall denote by CR the category of topological completely regular T_1 spaces and continuous mappings. CR

is complete, locally and colocally small category. The functor β which assigns to every space X from $\mathcal{CR}$ its Čech-Stone compactification, is an epireflector in $\mathcal{CR}$. Let us denote by $\mathcal{Comp}$ the corresponding subcategory of $\mathcal{CR}$. Further we denote by $\mathcal{Realcomp}$ the epireflective subcategory of all realcompact spaces. The realcompact reflection will be denoted, as usual, υX.

2.1. <u>Proposition</u>. Let $\mathcal{K}$ be a class of spaces from $\mathcal{CR}$ closed under continuous images, F an epireflector in $\mathcal{CR}$. Then $X \in \mathcal{K} * F$ if and only if for every embedding $j: Y \hookrightarrow X$ there exists $g: F(Y) \longrightarrow X$ such that $g\mu^Y = j$ (where $\mu^Y: Y \longrightarrow F(Y)$ is the corresponding reflection).

<u>Proof</u>: The necessity of the condition is evident. Let X from $\mathcal{CR}$ satisfy the condition. Let $Y \in \mathcal{K}$, $f: Y \to X$ continuous, and let $Z = f(Y)$. By assumption, $Z \in \mathcal{K}$; let $j: Z \hookrightarrow Y$ be the embedding. There exists $\bar{g}: F(Z) \to X$ continuous such that $\bar{g}\mu^Z = j$. Let $g = \bar{g}F(f)$. Then $g\mu^Y = \bar{g}F(f)\mu^Y = \bar{g}\mu^Z f = f$.
Hence, $X \in \mathcal{K} * F$.

2.2. <u>Corollary</u>. Be $\mathcal{K}$ a class of spaces from $\mathcal{CR}$ closed under continuous images. Then $X \in \mathcal{K} * \beta$ if and only if every subspace of X which lies in $\mathcal{K}$ is relatively compact.

2.3. <u>Example</u>: Let $\mathcal{Pseudocomp}$ denote the class of all pseudocompact spaces in $\mathcal{CR}$. We can easily see that $\mathcal{Pseudocomp}$ is exactly the class of all spaces fulfilling the condition $\upsilon X = \beta X$. Using 1.12′ we can see that $\mathcal{Pseudocomp}$ is a β-maxigenerator and that $\mathcal{Pseudocomp} * \beta = \beta\,\mathcal{hull}\,(\mathcal{Realcomp})$.

Let us suppose the existence of a measurable cardinal (i.e. a cardinal m such that there exists a set S of power m and a nontrivial two valued measure on $\exp S$ vanishing on onepoint sets). Let X be a discrete space of a measurable power. Then X is not realcompact, but every continuous mapping from a pseudocompact space into X can be extended to a continuous mapping into βX ; consequently $X \in \mathcal{P}seudocomp * \beta$.

Hence, the epireflective subcategory $\mathcal{R}ealcomp$ is distinct from its β-hull in the category $C\mathcal{R}$.

2.4. <u>Definition</u>. We shall call the subcategory $\mathcal{C}$ of $\mathcal{O}\!\mathcal{L}$ bireflective, if it is both epireflective and monocoreflective. (For example, the symmetric graphs in the category of all graphs.)

2.5. <u>Lemma</u>. Suppose $\mathcal{C}$ is bireflective in $\mathcal{O}\!\mathcal{L}$; let F_1 be the corresponding epireflector, F_2 the corresponding monocoreflector, $a, b \in |\mathcal{O}\!\mathcal{L}|$. Then $a \in \{b\} * F_1$ if and only if $b \in \{a\} - F_2$.

<u>Proof</u>: We denote by $\mu_1^a : a \longrightarrow F_1(a)$, $\mu_2^a : F_2(a) \longrightarrow a$ the morphisms generated by the functors F_1 , F_2 . Let $a \in \{b\} * F_1$. For every $(f : b \longrightarrow a) \in \mathcal{O}\!\mathcal{L}^m$ there exists $f_1 : F_1(b) \longrightarrow a$ such that $f_1 \mu_1^b = f$. $F_1(b) \in |\mathcal{C}|$, hence there exists exactly one $\vartheta : F_1(b) \longrightarrow F_2(a)$ such that $\mu_2^a \vartheta = f_1$. If we let $f' = \vartheta \mu_1^b$, then:

$\mu_2^a f' = \mu_2^a \vartheta \mu_1^b = f_1 \mu_1^b = f$. Hence, $b \in \{a\} - F_1$.

The converse implication is analogous.

The following three propositions are corollaries of the foregoing lemma:

2.6. Proposition. For every $K \subset |\mathcal{A}|$: $K * F_1$ is an F_2-maxigenerator, $K - F_2$ is an F_1-maxigenerator.

2.7. Proposition. For every $K_{F_1 max} = (K * F_1) - F_2$, $K_{F_2 max} = (K - F_2) * F_1$.

2.8. Proposition. Let $\mathcal{C}$ be an epireflective subcategory of $\mathcal{A}$, F the corresponding epireflector. Then $\mathcal{C}$ is bireflective if and only if every F -maxigenerator is a monocoreflective subcategory of $\mathcal{A}$. (Analogously the dual proposition.)

We note that in Haus (the category of Hausdorff spaces) or in CR or in $\mathcal{U}nif$, or in separated uniform spaces there is no bireflective subcategory except the whole category. The result for topology appears in [9]; the result for uniform spaces is due to M. Hušek (unpublished).

3.

We shall treat the applications of the theory in the category of uniform spaces now. By a uniform space we shall understand always a separated uniform space with the uniformity given by a system of uniform coverings (see [7]). We denote by $\mathcal{U}nif$ the category of separated uniform spaces and uniformly continuous mappings. The category $\mathcal{U}nif$ is complete and cocomplete.
The products are uniform products, equalisers embeddings of closed subspaces, coproducts are uniform sums, coequalisers

natural projections onto quotient spaces. The category $\mathcal{Unif}$ is evidently locally and colocally small.

We denote by $\mathcal{Fine}$ the monocoreflective subcategory of $\mathcal{Unif}$ consisting of all fine spaces; let α be the corresponding monocoreflector. Further, let $\mathcal{Subf}$ be the category of all subfine spaces (subspaces of fine ones). $\mathcal{Subf}$ is again monocoreflective; let ℓ be the corresponding coreflector. Let $\mathcal{Locf}$ be the category of all locally fine spaces (i.e. spaces, where every uniformly locally uniform covering is uniform). $\mathcal{Locf}$ is monocoreflective in $\mathcal{Unif}$; let λ be the corresponding coreflector. Then $\mathcal{Fine} \subset \mathcal{Subf} \subset \mathcal{Locf}$, i.e. $\alpha < \ell < \lambda$.

Further, let $\mathcal{Inj}$ be the class of all injective uniform spaces, $\mathcal{M}$ the class of metric spaces, $\gamma\mathcal{M}$ of complete metric spaces, $\mathcal{Compl}$ the class of all complete spaces. The last one is epireflective in $\mathcal{Unif}$. We denote, as usual, by γ the epireflector assigning to every uniform space its completion.

3.1. Proposition. Let $\mathfrak{X}$ be a class of uniform spaces closed under subspaces, F a monocoreflector in $\mathcal{Unif}$, $\mu^X : F(X) \longrightarrow X$ the corresponding monomorphisms. The uniform space X is from $\mathfrak{X} - F$ if and only if for every Y from $\mathfrak{X}$, $f: X \longrightarrow Y$ uniformly continuous and onto, there exists $g: X \longrightarrow F(Y)$ uniformly continuous so that $\mu^Y g = f$.

The proof is the dual analogon of the proof of 2.1.

3.2. Definition. We say that the reflector G in the category $\mathcal{Unif}$ is a modification, if for every uniform space μX the corresponding reflection $\mu X \longrightarrow G(\mu X)$ is an identity on X . Hence, $G(\mu)$ is the finest uniformity on X coarser than μ such that $G(\mu X)$ lies in the corresponding reflective subcategory. Analogously we define a comodification.

In [10] it is proved that in $\mathcal{Unif}$ every coreflector is a comodification.

3.3. Let Y be a uniform space, $\{M_a\}_{a \in J}$ all its metric uniformly continuous images, $p_a : \mu Y \longrightarrow M_a$ the corresponding mappings onto. Then μY is projectively generated by the family $\{p_a\}_{a \in J}$ (see for instance [7]). Let F be a comodification in the category $\mathcal{Unif}$ now. We denote $\mu^{(1)}\mu$ the uniformity on Y projectively generated by the family $\{Y \xrightarrow{p'_a} F(M_a)\}_{a \in J}$, where p'_a is, from the point of view of sets, the same as p_a . Further we define by transfinite induction:

$$\mu^{(\alpha+1)}\mu = \mu^{(1)}(\mu^{(\alpha)}\mu) \text{ , and}$$

if β is a limit ordinal, we set $\mu^{(\beta)}\mu = \bigvee_{\alpha < \beta} \mu^{(\alpha)}\mu$. There exists an ordinal number γ such that whenever $\delta \geq \gamma$ (δ an ordinal number), then $\mu^{(\delta)}\mu = \mu^{(\gamma)}\mu$. We set $\mu(\mu Y) = (\mu^{(\gamma)}\mu)Y$. We can easily see that μ is a functor in the category $\mathcal{Unif}$.

3.4. Proposition. The functor μ from the foregoing paragraph is exactly the monocoreflector $F_{\mathfrak{M}}$ (see 1.3) onto the subcategory $\mathfrak{M} - F$.

The proof follows easily from the definition of μ . The properties of the category $\mathfrak{M} - \alpha$ are described mainly in [4], further in [1],[2],[3],[13],[14]. I shall present some other examples here.

3.5. <u>Example</u>: $Inj - \alpha = Subf$.

<u>Proof</u>: Every uniform space X is embeddable into a product of injective uniform spaces $\prod_{\alpha \in J} Y_\alpha = Z$. Isbell showed in [7] that the uniformity on X is induced by embedding into the space αZ . Now, it is easy to complete the proof.

3.6. <u>Example</u>: $X \in Compl - \alpha$ if and only if its completion is a fine space.

<u>Proof</u>: Let $X \in Compl - \alpha$, and let $j : X \hookrightarrow \gamma X$ be the embedding into the completion. Then $j : X \longrightarrow \alpha \gamma X$ is uniformly continuous. Every Cauchy filter on $\alpha \gamma X$ is evidently Cauchy on γX . Hence, $\alpha \gamma X$ is complete. Then (γ is a reflector) γX is isomorphic to $\alpha \gamma X$. Hence, γX is fine.
Conversely, let γX be a fine space, Y a complete space, $f : X \longrightarrow Y$ uniformly continuous, $j : X \hookrightarrow \gamma X$ the embedding. There exists (exactly one) $h : \gamma X \longrightarrow Y$ uniformly continuous such that $hj = f$.
γX is fine so that h is uniformly continuous into αY . Hence, f is uniformly continuous from X to αY , so $X \in$ $\in Compl - \alpha$.

3.7. <u>Example</u>: $\gamma \mathcal{M} - \alpha = \gamma \mathcal{M} - \ell = \gamma \mathcal{M} - \lambda$.

<u>Proof</u>: Ginsburg and Isbell proved (see [7]) that for every complete metric space φM , $\lambda \varphi M = \alpha \varphi M$. Further $\alpha < \ell < \lambda$, from which the proposition follows immediately.
From the last example it follows that $Subf \subset \gamma \mathcal{M} - \alpha$. Hence, $(\gamma \mathcal{M})_{\alpha max} \subset (Inj)_{\alpha max}$.

3.8. <u>Definition</u>. Let $\mathcal{C}$ be a full subcategory of *Unif*. We define the full subcategory $\mathit{Sub}-\mathcal{C}$ of *Unif*: $X \in \mathit{Sub}-\mathcal{C}$ if and only if there exists some Y in $\mathcal{C}$ such that X is uniformly embeddable into Y.

3.9. <u>Theorem</u>. Let $\mathcal{C}$ be a coreflective subcategory in *Unif*, F the corresponding coreflector. Then $\mathit{Sub}-\mathcal{C}$ is monocoreflective. (Let us denote $\ell_{\mathcal{C}}$ the corresponding coreflector.)

<u>Proof</u>: Let $X \in |\mathit{Unif}|$. We shall construct $\ell_{\mathcal{C}} X$. There exists an injective space Y and $j: X \hookrightarrow Y$ embedding. We define $\ell_{\mathcal{C}} X$ to be the set X with the uniformity induced by the embedding $j': X \hookrightarrow F(Y)$. Clearly, $X \in \mathit{Sub}-\mathcal{C}$. Let $Z \in \mathcal{C}$, $i: X' \hookrightarrow Z$ embedding, $f: X' \longrightarrow X$ uniformly continuous. Then there exists $f': Z \longrightarrow Y$ such that $f'i = jf$. f' is uniformly continuous into $F(Y)$, $f' \restriction X'$ is uniformly continuous into $\ell_{\mathcal{C}} X$. The unicity is evident. Hence, $\ell_{\mathcal{C}}$ is a comodification.
If Y is injective, then evidently $\ell_{\mathcal{C}} Y = F(Y)$. So we get a stronger proposition: $\mathit{Sub}-\mathcal{C} = \mathit{Inj}-F$.

3.10. <u>Example</u>: $\mathit{Sub}-(\mathit{Compl}-\alpha) = \mathit{Subf}$.

The proof is evident.

<u>Corollary</u>: $\mathit{Inj}-\alpha_{\mathit{Compl}} = \mathit{Inj}-\ell = \mathit{Inj}-\alpha$.

3.11. <u>Proposition</u>. Let F be a coreflector in *Unif*, $\mathcal{L}$ the corresponding monocoreflective subcategory. $\mathcal{L}$ is closed under subspaces if and only if $(\mathit{Inj})_{F\max} = \mathit{Unif}$.

The proof is evident.

Let $\mathcal{K} - F$ be closed under subspaces (in $\mathcal{U}nif$), then $\mathcal{K} - F = \mathcal{I}nj - F_K \supset \mathcal{I}nj - F$, so there is $(\mathcal{K})_{F max} \subset$ $\subset (\mathcal{I}nj)_{F max}$; in particular $\mathcal{K} \subset (\mathcal{I}nj)_{F max}$.

3.12. Lemma. Let F be a comodification in $\mathcal{U}nif$, $\mathcal{L}$ the corresponding coreflective subcategory. $\mathcal{L}$ is closed under subspaces if and only if for every $X, Y \in |\mathcal{U}nif|$, $X \subset Y$ implies $F(X) \subset F(Y)$.

The proof follows immediately from the fundamental properties of comodifications.

3.13. Theorem. Let F, G be two comodifications in $\mathcal{U}nif$ preserving uniform embeddings. Then the class $\mathcal{K} =$ $= \{X \mid F(X) = G(X)\}$ is closed under subspaces.
The theorem is an easy consequence of the lemma 3.12 .

3.14. Corollaries: 1) The class $\{X \mid \ell X = \lambda X\}$ is hereditary.
2) Let $\mathcal{L}$ be monocoreflective, F the corresponding coreflector. Then, if $\mathcal{L}$ is hereditary, $\mathcal{K} - F$ hereditary, then there is $\mathcal{K}_{F max}$ hereditary.
3) Since for every complete metric space X , $\ell X = \lambda X$, it follows from 1) that $\mathcal{M} \subset \{X \mid \ell X = \lambda X\}$; hence, for every metric space M , $\ell M = \lambda M$; and $\mathcal{M} - \ell = \mathcal{M} - \lambda$.

3.15. Theorem. $\gamma \mathcal{M} - \alpha \subset \mathcal{M} - \ell$.

Proof: Let $X \in \gamma \mathcal{M} - \alpha$, $M \in \mathcal{M}$, $f : X \longrightarrow M$ uniformly continuous, $j : M \hookrightarrow \gamma M$ embedding. Further, let $i : \alpha \gamma M \longrightarrow \gamma M$ be the uniformly continuous identity mapping, $\iota : \ell M \longrightarrow M$ the identity mapping (uniformly continuous). There exists $g' : X \longrightarrow \alpha\gamma M$ uniformly continuous such that $i g' = j f$. The functor ℓ commutes with completion [3], so that $\alpha\gamma M = \ell\gamma M = \gamma\ell M$. Hence, $g' = j' g$, where $g : X \longrightarrow \ell M$ is uniformly continuous, $j' : \ell M \hookrightarrow \alpha\gamma M$ embedding. Evidently, $i j' = j\iota$, $i j' g = j \iota g = j f$; j is a monomorphism, so $\iota g = f$. Consequently $X \in \mathcal{M} - \ell$.

3.16. <u>Corollary</u>: From 3.7, 3.14, 3.15 we get:

$$\gamma\mathcal{M} - \alpha = \mathcal{M} - \ell = \gamma\mathcal{M} - \ell = \mathcal{M} - \lambda = \gamma\mathcal{M} - \lambda .$$

Let us denote by $\mathcal{Precomp}$ the category of precompact uniform spaces and uniformly continuous mappings. $\mathcal{Precomp}$ is epireflective in $\mathcal{Unif}$. The corresponding reflection (it is a modification) will be denoted p. Notice that, whenever μ is a uniformity on X, $p\mu$ is topologically compatible with μ. See [7].

3.17. <u>Example</u>: $\mathcal{Precomp} - \alpha = \mathcal{Fine}$.

<u>Proof</u>: For every uniform space μX, $\alpha\mu X = \alpha p \mu X$. The identity $id : \mu X \longrightarrow p\mu X$ is a precompact reflection. If $\mu X \in \mathcal{Precomp} - \alpha$, then the identity $id : \mu X \longrightarrow \alpha p \mu X = \alpha\mu X$ is uniformly continuous. Hence, $\alpha\mu = \mu$, so μX is fine. The converse inclusion is trivial.

3.18. <u>Example</u>: $\mathcal{Subf} - \alpha = \mathcal{Fine}$.

<u>Proof</u>: Let X be a precompact space. Then $\gamma X = \gamma p X$ is a Samuel compactification of X, and is a fine space. Then X is subfine; consequently $\mathcal{Precomp} \subset \mathcal{Subf}$. Using 1.2 we get $\mathcal{Subf} - \alpha \subset \mathcal{Precomp} - \alpha$ ($= \mathcal{Fine}$). Hence, $\mathcal{Subf} - \alpha = \mathcal{Fine}$.

3.19. Corollaries: Whenever $\mathcal{Subf} \subset \mathcal{X}$, then $\mathcal{X} - \alpha =$ $= \mathcal{Fine}$. For example:

$\mathcal{Locf} - \alpha = \mathcal{Fine}$,

$(\gamma \mathcal{M} - \alpha) - \alpha = \mathcal{Fine}$.

By 3.6, $\mathcal{Precomp} \subset \mathcal{Compl} - \alpha$, hence $(\mathcal{Compl} - \alpha) - \alpha =$ $= \mathcal{Fine}$.

References: 3.5 is stated in [3] and attributed to Isbell - Rice. It appears also in [12]. 3.6 appears also in [5]. 3.4 appears also in [13] and [14]. In the special case, for $F = \alpha$, $F_{\mathcal{M}} = \mu^{(1)}$ - see [1],[2],[13],[14] (independently due to Frolík and Rice).

All the overlapping results were obtained independently.

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