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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 2, 295--303

Persistent URL: http://dml.cz/dmlcz/105492

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14,2 (1973)

QF-3' MODULES AND RINGS

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Abstract: Some properties of pseudo-injective modules and self-pseudo-injective rings were studied in [11]. The last notion appears in the literature (see e.g. [5], [6], [9], [10]) also as the QF-3' rings. Jans [6] has characterized these rings in terms of preradicals. In this paper the pseudo-injective modules will be called QF-3' modules and will be characterized by using preradicals. Further, the characterization of QF-3' rings as endomorphism rings of some modules is presented and the QF-3' modules over such rings are investigated. Some results concerning Morita equivalence of QF-3' modules and rings appears as corollaries.

Key words: Preradical, idempotent preradical, torsion preradical, radical, QF-3' module, QF-3' ring, flat module, endomorphism ring, Morita equivalence.

AMS, Primary: 16 A36, 18E40 Ref.Ž. 2.723.2

Secondary: 16 A50

All the rings considered below will be associative with identity and all modules will be unitary. The category of left (right) R-modules is denoted by ${}_R\mathcal{M}$ ($\mathcal{M}_R$) and ${}_R M$ (M_R, ${}_R M_S$ respectively) means M is a left R-module (right R-module, R-S-bimodule respectively). When no confusion can arise, by a word module we shall always mean an unitary left R-module.

A preradical φ for ${}_R\mathcal{M}$ is any subfunctor of the identity, i.e. φ assigns to each module M a submodule $\varphi(M)$

in such a way that every homomorphism $M \longrightarrow N$ induces $\varphi(M) \longrightarrow \varphi(N)$ by restriction. A preradical φ is said to be idempotent if $\varphi^2 = \varphi$, torsion if φ is left exact and is called a radical if $\varphi(M/\varphi(M)) = 0$. It is well-known that φ is torsion iff $L \subseteq M$ implies $\varphi(L) = L \cap \varphi(M)$ (see e.g. [8], Prop.1.4). For a preradical φ, a module M is called φ-torsion if $\varphi(M) = M$ and φ-torsion-free if $\varphi(M) = 0$. It is known that an idempotent radical φ is torsion iff the class of φ-torsion-free modules is closed under taking injective envelopes (see e.g. [7], Pro.2.9). The injective envelope of a module M will be denoted by $\hat{M}$ and $Z(M)$ is the singular submodule of M. For the homological notions and results we refer to [3].

For $M, N \in {}_R\mathcal{M}$ let us define $\varphi_M(N) = \bigcap_{f \in Hom_R(N,M)} Ker\, f$.

1. <u>Lemma</u>. For every module $M \in {}_R\mathcal{M}$, φ_M is a radical (not necessarily idempotent).

Proof: For $\varphi \in Hom_R(N,K)$, $g \in Hom_R(K,M)$ and $x \in \varphi_M(N)$ we have $x\varphi g = 0$, hence $x\varphi \in \varphi_M(K)$ and φ_M is a preradical. For $x \in N \dot{-} \varphi_M(N)$ there exists $f \in Hom_R(N,M)$ with $xf \neq 0$. Since $\varphi_M(N) \subseteq Ker\, f$, f induces $\bar{f} \in Hom_R(N/\varphi_M(N), M)$ with $(x + \varphi_M(N))\bar{f} = xf \neq 0$ and $N/\varphi_M(N)$ is therefore φ_M-torsion-free.

2. <u>Definition</u>. A module $N \in_R \mathcal{M}$ is said to be M-torsion-less if $\varphi_M(N) = 0$.

3. <u>Definition</u>. A module $M \in_R \mathcal{M}$ is said to be a

QF-3' module if $\hat{M}$ is M-torsion-less.

A ring R is a left QF-3' ring if ${}_R R$ is a QF-3'-
- module.

For two preradicals φ, σ we shall write $\varphi \leq \sigma$ if $\varphi(M) \subseteq \sigma(M)$ for every $M \in {}_R\mathcal{M}$. It is a well-known fact that any preradical φ contains a unique largest idempotent preradical which we denote by $\overline{\varphi}$ (see e.g.[2],[8]).

Generalizing the ideas of Jans [6] we obtain the following results:

4. <u>Proposition</u>. The following conditions are equivalent for a module $M \in {}_R\mathcal{M}$:

(1) $\varphi_M = \overline{\varphi}_M$;

(2) φ_M is idempotent;

(3) the class of M-torsion-less modules is closed under extensions.

Proof: (2) $\Longrightarrow$ (3). Let $0 \longrightarrow K \xrightarrow{\alpha} L \xrightarrow{\beta} N \longrightarrow 0$ be a short exact sequence with K, N M-torsion-less and let φ_M be idempotent. Now $(\varphi_M(L))\beta \subseteq \varphi_M(N) = 0$ yields $\varphi_M(L) \subseteq \operatorname{Im}\alpha$ and $\varphi_M(L) = \varphi_M^2(L) \subseteq \varphi_M(\operatorname{Im}\alpha) = \varphi_M(K) = 0$ gives L M-torsion-less.

(3) $\Longrightarrow$ (2). By Lemma 1, $N/\varphi_M(N)$ and $\varphi_M(N)/\varphi_M^2(N)$ are M-torsion-less, so that $N/\varphi_M^2(N)$ is M-torsion-less by hypothesis. But $\varphi_M(N/\varphi_M^2(N)) = \varphi_M(N)/\varphi_M^2(N)$ by [8], Lemma 1,2 which shows φ_M is idempotent.

The equivalence of (1) and (2) is obvious.

5. <u>Proposition</u>. Let $M \in {}_R\mathcal{M}$. The class of φ_M -torsion modules is closed under submodules iff $\overline{\varphi}_M = \varphi_{\hat{M}}$.

Proof: Let the class of φ_M -torsion modules be closed under submodules. It is easy to see that $\varphi_{\hat{M}}$ is a torsion radical and therefore $\varphi_{\hat{M}} \leq \overline{\varphi}_M$ owing to the definition of $\overline{\varphi}_M$. Suppose $\overline{\varphi}_M(N) = N$ and $0 \neq f \in Hom_R(N, \hat{M})$. Then f induces a non-zero homomorphism $f' : N' = Mf^{-1} \longrightarrow M$ which contradicts to $\overline{\varphi}_M(N') = N'$. Hence $\overline{\varphi}_M$ and $\varphi_{\hat{M}}$ have the same classes of torsion modules and $\overline{\varphi}_M = \varphi_{\hat{M}}$ by [2], Prop. 1.

The converse follows immediately from the fact that φ_M and $\overline{\varphi}_M$ have the same classes of torsion modules and that $\varphi_{\hat{M}}$ is a torsion radical.

6. <u>Theorem</u>. The following conditions for a module $M \in {}_R\mathcal{M}$ are equivalent:

(1) M is a QF - 3' module;

(2) $\varphi_M = \varphi_{\hat{M}}$;

(3) φ_M is torsion.

Proof: (1) $\Longrightarrow$ (2). Let $x \in \varphi_M(N)$ and $f \in Hom_R(N, \hat{M})$, $xf \neq 0$. Since $\varphi_M(\hat{M}) = 0$, there exists $g \in Hom_R(\hat{M}, M)$ with $xfg \neq 0$ contradicting to $x \in \varphi_M(N)$. Hence $\varphi_M(N) \subseteq \varphi_{\hat{M}}(N)$ and $\varphi_M = \varphi_{\hat{M}}$, the inverse inclusion being obvious.

(2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). We have $0 = \varphi_M(M) = M \cap \varphi_M(\hat{M})$, so that

$\rho_M(\hat{M}) = 0$, M being essential in $\hat{M}$.

7. <u>Theorem</u>. Let R be a ring and $M \in {}_R\mathcal{M}$ with $Z(M) = 0$. Then ρ_M is torsion iff $\overline{\rho}_M$ is so.

Proof: If ρ_M is torsion, then $\rho_M = \rho_{\hat{M}} \leq \overline{\rho}_M \leq \rho_M$ by 6, and $\overline{\rho}_M = \rho_M$ is torsion.

Conversely, let $0 \neq K = \rho_M(\hat{M})$. For $\rho_M(\hat{K}) = \hat{K} = \overline{\rho}_M(\hat{K})$ we have $\overline{\rho}_M(M \cap K) = M \cap K = \rho_M(M \cap K)$ by hypothesis and hence $M \cap K = 0$ contradicting to the essentiality of M in $\hat{M}$. We can therefore take $0 \neq f \in \in \mathrm{Hom}_R(\hat{K}, M)$ and $x \in \hat{K}$ with $xf \neq 0$. Since $Z(M) = 0$, $(0 : xf) = \{r \in R,\ rxf = 0\}$ is not essential in R and $(0 : xf) \cap L = 0$ for some non-zero left ideal L of R . Now $Lx \cap K \neq 0$ since K is essential in $\hat{K}$, so that there exists $r \in L$ such that $rx \in K$ and $rxf \neq 0$.

Finally, f can be extended to an element of $\mathrm{Hom}_R(\hat{M}, M)$, since $\hat{K}$ is a direct summand of $\hat{M}$, which contradicts to the definition of K .

8. <u>Corollary</u>. (Vinsonhaler [10], Prop.2.) Let R be a ring with $Z(R) = 0$. If the class of modules with zero duals is closed under submodules, then R is a QF-3' ring.

9. <u>Theorem</u>. Let Q be a QF-3' module and T a module. If $\rho_Q(T) = 0$ then $Q \oplus T$ is a QF-3' module. Conversely, if $\rho_Q(\hat{T}) \subseteq \rho_T(\hat{T})$ and $Q \oplus T$ is a QF-3' module, then $\rho_Q(T) = 0$.

Proof: The isomorphism $\mathrm{Hom}_R(M, Q \oplus T) \cong$
$\cong \mathrm{Hom}_R(M, Q) \oplus \mathrm{Hom}_R(M, T)$ shows that $\varphi_{Q\oplus T} = \varphi_Q \cap$
$\cap\, \varphi_T$. Hence $\varphi_{Q\oplus T}(\hat{Q} \oplus \hat{T}) = (\varphi_Q(\hat{Q}) \oplus \varphi_Q(\hat{T})) \cap \varphi_T(\hat{Q} \oplus \hat{T})$.
By hypothesis, $\varphi_Q(\hat{Q}) = \varphi_Q(\hat{T}) = 0$ (φ_Q is torsion by 6) showing $Q \oplus T$ is a $QF-3'$ module.

Conversely, the same equality gives $0 = \varphi_{Q\oplus T}(\hat{Q} \oplus \hat{T}) =$
$= \varphi_Q(\hat{T}) \cap (\varphi_T(\hat{Q}) \oplus \varphi_T(\hat{T})) = \varphi_Q(\hat{T}) \cap \varphi_T(\hat{T}) = \varphi_Q(\hat{T})$ and hence $\varphi_Q(T) = 0$.

10. <u>Corollary</u> (Zuckerman [11],Th.1). Let R be a (left) noetherian hereditary ring and $A = Q \oplus T$ a left R-module where Q is injective and T reduced. Then A is a $QF-3'$-module iff $\varphi_Q(T) = 0$.

Proof: There is $\varphi_T(\hat{T}) = \hat{T}$ over a left hereditary ring.

11. <u>Theorem</u>. Let $M \in_R \mathcal{M}$ be a module which is flat as a right module over its endomorphism ring S. If $N \in_R \mathcal{M}$ is a $QF-3'$ module then the left S-module $\mathrm{Hom}_R(M,N)$ is a $QF-3'$ module.

Proof: For an exact sequence $0 \longrightarrow {}_S A \xrightarrow{\alpha} {}_S B$ we have $0 \longrightarrow M \otimes_S A \longrightarrow M \otimes_S B$ exact by flatness of M_S.
Hence the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_S({}_S B, \mathrm{Hom}_R(M,\hat{N})) & \xrightarrow{\alpha} & \mathrm{Hom}_S({}_S A, \mathrm{Hom}_R(M,\hat{N})) \\
\wr\| & & \wr\| \\
\mathrm{Hom}_R(M \otimes_S B, \hat{N}) & \longrightarrow & \mathrm{Hom}_R(M \otimes_S A, \hat{N}) \longrightarrow 0
\end{array}$$

in which the verticals are natural isomorphisms, shows α^* is an epimorphism and $Hom_R(M, \hat{N})$ is an injective S-module.

Now for $0 \neq \alpha \in Hom_R(M, \hat{N})$ we have $m\alpha = x \neq 0$ for some $m \in M$. Since ${}_RN$ is a $QF-3'$ module, there exists $f \in Hom_R(\hat{N}, N)$, $xf \neq 0$. Now $\alpha f_* \neq 0$ since $m \alpha f_* = xf \neq 0$ showing $Hom_R(M, \hat{N})$ is $\rho_{Hom_R(M,N)}$-torsion-less. The S-injective envelope of $Hom_R(M, N)$ is therefore $Hom_R(M, N)$-torsion-less as a submodule of $Hom_R(M, \hat{N})$ and we are ready.

12. <u>Theorem</u>. Let ${}_RM$ be a $QF-3'$ module which is flat as a right module over its endomorphism ring S. Then S is a $QF-3'$ ring.
Conversely, every $QF-3'$ ring can be obtained in such a way.

Proof: The direct part follows from 11 immediately while the converse is trivial.

13. <u>Corollary</u> (Tachikawa [9], Prop.1.1). Every quotient ring of a $QF-3'$ ring R is $QF-3'$.

Proof: For $M \subseteq M' \subseteq \hat{M}$, M $QF-3'$ we have M' is $QF-3'$ since $\rho_M \geq \rho_{M'} \geq \rho_{\hat{M}'} = \rho_{\hat{M}} = \rho_M$. Now the corollary follows from 12 and the well-known fact that every quotient ring is the endomorphism ring of some R-module between R and $\hat{R}$.

14. <u>Remark</u>: It should be remarked that the condition

M_S is flat cannot be dropped in general. For example, the quasi-cyclic p-group $C(p^\infty)$ is a QF-3' Z-module (since it is injective) and its endomorphism ring is the ring of p-adic integers which is not QF-3'. Of course, $C(p^\infty)$ is torsion and hence not flat over the ring of p-adic integers.

15. Theorem. Let R and S be Morita equivalent rings via $T = Hom_R(P,-)$. If M is a QF-3' left R-module then $T(M)$ is a QF-3' left S-module.

Proof: Follows immediately from 11 since P_S is projective (see [1], Ch.II, §§ 3, 4).

16. Corollary. Let R and S be Morita equivalent rings via $T = Hom_R(P,-)$. Then T induces a one-to-one correspondence between the isomorphism classes of QF-3' R-modules and QF-3' S-modules.

Proof: Let ${}_SM'$ be a QF-3' S-module. Then it follows from the well-known properties (see [1], Ch.II, §§ 3, 4) of equivalences of categories that ${}_SM' \cong T({}_RM)$, $\widehat{{}_SM'} \cong T(\widehat{{}_RM})$ and $\widehat{{}_SM'} \hookrightarrow \prod_{\alpha\in\Lambda} M'_\alpha$, $M'_\alpha \cong M'$, gives

$$\widehat{{}_RM} \cong P \otimes_S T(\hat{M}) \hookrightarrow P \otimes_S T(\prod_{\alpha\in\Lambda} M_\alpha) \cong \prod_{\alpha\in\Lambda} M_\alpha \,, \; M_\alpha \cong M \,,$$

showing ${}_RM$ is a QF-3' R-module. Now it suffices to use 15.

17. Corollary. Let R and S be Morita equivalent rings. Then R is QF-3' iff S is so.

R e f e r e n c e s

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(Oblatum 8. 5. 1973)