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Lubomír Vašák

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GENERALIZED MARKUŠEVIČ BASES

L. VAŠÁK, Praha

Abstract:

Necessary and sufficient condition for $w^* - w$-continuity of the Dyer embedding of a B-space X^* with a generalized Markuševič basis to $C_0(H)$ is found.

Key words:

Markuševič basis, weakly compactly generated Banach spaces.

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Notations. In this paper, X denotes a real B-space with the norm $\|\cdot\|$, X^* its dual B-space with the w^*-topology w^*, and X^{**} its second dual, $\varkappa$ denotes the natural embedding of X into X^{**}.

For each locally compact topological space H we denote by $C_0(H)$ the B-space of all real continuous functions on H which vanish at infinity with the supremum norm $|||\cdot|||$ and the weak topology w. In the case H is discrete we denote $C_0(H) = c_0(H)$.

For any normed linear space V and any set $M \subset V$ we denote by $\mathrm{lin}\, M$ the linear span of M in V and

$sp\, M$ the norm-closure of $lin\, M$.

If Γ is a set, $\Gamma \neq \emptyset$, then $\ell_\infty(\Gamma)$ is the B - space of all bounded mappings from Γ into R, R being real numbers.

Definition 1 (see [2]). A B -space X has a generalized Markuševič basis (shortly gMb) (B, γ), $B \subset X$, $\gamma \subset X^*$ if:

(i) if $b \in B$, $f \in \gamma$, then $b \neq 0$, $f \neq 0$ and $f(b) \in \{0, 1\}$,

(ii) for any finite set $D \subset B$ there exists a finite set $D' \subset B$ so that: $D \subset lin\, D'$ and if $f \in \gamma$, then

$$f\left(\sum_{d \in D'} d\right) \in \{0, 1\} ,$$

(iii) $sp\, B = X$,

(iv) γ is total over X and γ is bounded in X^* .

Remark. (a) Each finite set $D \subset B$ such that for any $f \in \gamma$, $f\left(\sum_{d \in D} d\right) \in \{0, 1\}$, is called γ-orthogonal set.

(b) Markuševič basis (shortly Mb) is a gMb for which (B, γ) is a biorthogonal collection.

(c) From the condition (ii) it is easily seen that $lin\, B = \{\sum_{j=0}^{m} \lambda_j d_j ; \{d_0, \dots, d_m\}$ is a γ-orthogonal subset of B , $\{\lambda_0, \dots, \lambda_m\} \subset R$, $m \in \omega\}$. We will call each such $\sum_{j=0}^{m} \lambda_j d_j$ (B, γ) -combination.

(d) Evidently any Mb is a gMb. It was proved in [4] that any weakly compactly generated B -space X (i.e. there is a weakly compact $K \subset X$ such that $X = sp\, K$) has a Mb. The following two examples show that there is a great difference between Mb's and gMb's and the class of all B -spaces with a gMb is very large.

Example 1. Let Γ be a nonempty set. Then $\ell_\infty(\Gamma)$ has a gMb.

Proof. Let $\gamma = \{\varphi_\alpha ; \alpha \in \Gamma\}$ where $\varphi_\alpha(x) = x(\alpha)$ for any $x \in \ell_\infty(\Gamma)$, $\alpha \in \Gamma$. It is easily seen that for $B = \{x \in \ell_\infty(\Gamma); x(\alpha) \in \{0,1\}$ for any $\alpha \in \Gamma$; $x \neq 0\}$ (B, γ) satisfies Conditions (i),(iii),(iv) from Definition 1. We show that (B, γ) satisfies (ii):

Let $\{b_0, \dots, b_n\} \subset B$, $n \in \omega$. For any set $A \subset \{0,\dots,n\}$, we define $d_A \in \ell_\infty(\Gamma)$ so that $d_A(\alpha) = 1$ iff $A = \{i \in \{0,\dots,n\}; b_i(\alpha) = 1\}$, $d_A(\alpha) = 0$ in all other cases, for any $\alpha \in \Gamma$. Then a finite set $\{d_A ; \emptyset \neq A \subset \{0,\dots,n\}\} \subset$ $\subset B$ is γ-orthogonal and for any $i \in \{0,\dots,n\}$

$$b_i = \sum \{d_A ; \emptyset \neq A \subset \{0,\dots,n\}, i \in A\} .$$

Example 2. It is easy to see that if (B, γ) is a Mb and $B', B'' \subset B$ so that $B' \cup B'' = B$, $B' \cap B'' = \emptyset$, then $sp\, B' \cap sp\, B'' = \{0\}$.

The following example shows that in the case of gMb this is far from true:

In the space $c_0(\Gamma)$, Γ a nonempty set, let $\gamma =$ $= \{e_\alpha, \alpha \in \Gamma\}$, where $e_\alpha \in \ell_1(\Gamma)$ so that $e_\alpha(\beta) = 1$ if $\alpha = \beta$, $e_\alpha(\beta) = 0$ if $\alpha \neq \beta$ for any $\alpha, \beta \in \Gamma$. Let $B = \{x \in c_0(\Gamma); x(\alpha) \in \{0,1\}$ for any $\alpha \in \Gamma$, $x \neq 0\}$ .Then (B, γ) is a gMb of $c_0(\Gamma)$. Suppose that $B = B' \cup B''$, $B' \cap B'' = \emptyset$ and $sp\, B' \cap sp\, B'' = \{0\}$. We show that then $B' = \emptyset$ or $B'' = \emptyset$.

We choose some $\alpha_0 \in \Gamma$. Let $e_{\alpha_0} \in B'$. Let $\alpha \in \Gamma \setminus \{\alpha_0\}$.

If $e_\alpha \in B''$ then $e_\alpha + e_{\alpha_0} \in B$ can be neither in B' nor in B'', because in both cases $sp\, B' \cap sp\, B'' \neq \{0\}$. Therefore $e_\alpha \in B'$, hence $B \subset lin\, B'$ i.e. $B'' = \emptyset$.

Definition 2 (Dyer,[2], p.53-55). Let a B-space X have a gMb (B, γ). Let $\gamma' = \gamma^* \setminus \{0\}$, where γ^* is the w^*-closure of γ in X. Then (B, γ') is a gMb of X, γ'-orthogonality is γ-orthogonality and γ' endowed with the partialized w^*-topology from X^* is a Boolean space which we denote by H. We call the Dyer embedding of X into $C_0(H)$ the mapping $T: X \longrightarrow C_0(H)$ defined $Tx(f) = f(x)$ for any $f \in \gamma'$, $x \in X$. Dyer shows in [2] that T is continuous, linear, one-to-one and onto a dense subset of $C_0(H)$.

Remark 2. If (B, γ) is a Mb, then $C_0(H) = c_0(H)$ and so X has an equivalent strictly convex norm (see [1]). Because $\ell_\infty(\Gamma)$ for Γ infinite has no equivalent strictly convex norm ([1]), $\ell_\infty(\Gamma)$ is an example of a B-space which has a gMb and has no Mb. Our aim is to find the necessary and sufficient condition for $w^* - w$-continuity of the Dyer embedding of a B-space X^* with a gMb into $C_0(H)$. We need the following lemmas and definitions.

Definition 3. Suppose X has a gMb (B, γ). Then we denote by $\ell_0(B)$ the following B-space:
$\ell_0(B) = \{y \in \ell_\infty(B)$; (a) there exist $C > 0$ so that for any finite γ-orthogonal set $D \subset B$, $\sum_{d \in D} |y(d)| \leq C$; (b) if $\sum_{i=1}^{n} \lambda_i b_i$ and $\sum_{j=0}^{m} \mu_j d_j$ are two (B, γ)-combinations and $\sum_{i=0}^{n} \lambda_i b_i = \sum_{j=0}^{m} \mu_j d_j$, then $\sum_{i=0}^{n} \lambda_i y(b_i) =$

$= \sum_{j=0}^{m} \mu_j y(d_j)\}$ as the vector subspace of $\ell_\infty(B)$ and with the norm $\|\cdot\|_0$ defined $\|y\|_0 = \sup\{\sum_{d\in D} |y(d)|$; D is a finite γ-orthogonal subset of $B\}$ for any $y \in \ell_0(B)$.

<u>Lemma 1.</u> $\ell_0(B)$ is a B-space.

<u>Proof</u>: Easy.

<u>Lemma 2.</u> $\ell_0(B)$ is isometrically isomorphic with $[C_0(H)]^*$.

<u>Proof.</u> Let $A = \{\sum_{i=0}^{n} \lambda_i T(b_i)$; $\sum_{i=0}^{n} \lambda_i b_i$ is a (B,γ)-combination$\}$, where T is the Dyer embedding of X into $C_0(H)$, A is a normed linear space (with the norm $|||\cdot|||$) with the same dual as $C_0(H)$. For any $y \in \ell_0(B)$, $x \in A$ we define $\langle x, y\rangle = \sum_{j=0}^{n} \lambda_i y(b_i)$, where $x = \sum_{i=0}^{n} \lambda_i T(b_i)$, $\sum_{i=0}^{n} \lambda_i b_i$ is a (B,γ)-combination. The definition of $\langle\cdot,\cdot\rangle$ is correct because if $\sum_{i=0}^{m} \lambda_i b_i$ and $\sum_{j=0}^{m} \mu_j d_j$ are two (B,γ)-combinations, then: $\sum_{i=0}^{n} \lambda_i T(b_i) = \sum_{j=0}^{m} \mu_j T(d_j)$ implies $\sum_{i=0}^{n} \lambda_i b_i = \sum_{j=0}^{m} \mu_j d_j$ and so $\sum_{i=0}^{n} \lambda_i y(b_i) = \sum_{j=0}^{m} \mu_j y(d_j)$ as follows from the condition (b). We show that $\langle A, \ell_0(B)\rangle$ is a dual pair and that in this duality $\ell_0(B)$ represents all elements of A^*, where A^* is the dual of A. By means of (b) and properties of T we show that $\langle\cdot,\cdot\rangle$ is bilinear. Obviously $\langle\cdot,\cdot\rangle$ is separated.

Let $y \in \ell_0(B)$, $x \in A$, $x = \sum_{i=0}^{m} \lambda_i T(b_i)$, $\sum_{i=0}^{n} \lambda_i b_i$ is the (B,γ)-combination. Then:

$$|\langle x, y\rangle| = |\sum_{i=0}^{m} \lambda_i\, y(b_i)| \leq \max_{i=1,\dots,m} |\lambda_i| \cdot \sum_{i=0}^{m} |y(b_i)| \leq$$
$$\leq |||x||| \cdot \|y\|_0$$

because $|||x||| = \max |\lambda_i|$ as it is easily seen from the fact that $T(b_i) = \chi_{E(b_i)}$, where $E(b_i) = \{f \in \gamma'; f(b_i) = 1\}$, $i = 0, 1, \dots, m$, are pairwise disjoint and $\chi_{E(b_i)}$ denotes the characteristic function of $E(b_i)$. Hence any $y \in \ell_0(B)$ represents the unique $\mu_y \in A^*$. Moreover, it is easy to prove that $|||\mu_y|||_* = \|y\|_0$, where $|||\cdot|||_*$ is the dual norm in A^*.

Let $\mu \in A^*$. For any $b \in B$ we define $y(b) = \mu(Tb)$. Then y evidently satisfies (b) and if $\{b_0, \dots, b_n\} \subset B$ is γ-orthogonal, then

$$\sum_{i=0}^{n} |y(b_i)| = \sum_{i=0}^{n} |\mu(Tb_i)| = \sum_{i=0}^{n} a_i \cdot \mu(Tb_i) =$$
$$= \mu(\sum_{i=0}^{n} a_i T b_i) \leq |||\mu|||_*, \text{ where } a_i = \operatorname{sign} \mu(Tb_i),$$

for any $i = 0, 1, \dots, n$. Hence $y \in \ell_0(B)$. Obviously $\mu = \mu_y$. Thus A^* is isometrically isomorphic with $\ell_0(B)$.

Theorem. Let a B-space X^* have a gMb (B, γ), let T be the Dyer embedding of X^* into $C_0(H)$. Then T is w^*-w-continuous iff each $F \in X^{**}$ satisfying the condition:

(1) $\sup\{\sum_{d \in D} |F(d)|;\ D \subset B \text{ is } \gamma\text{-orthogonal}\} < +\infty$

is an element of $\varkappa(X)$.

Proof. For any $\mu \in A^*$ we denote by $\tilde{\mu}$ the continuous extension of μ on $C_0(H)$.

T is w^*-w-continuous iff for any $\mu \in A^*$ it holds

$\tilde{\mu}\ T \in \varkappa(\mathcal{X})$. From Lemma 2 it follows that T is w^*-w-continuous iff for any $y \in \ell_0(B)$ $\tilde{\mu}_y T \in \varkappa(\mathcal{X})$.

a) Suppose that for any $y \in \ell_0(B)$, $\tilde{\mu}_y T \in \varkappa(\mathcal{X})$. Let $F \in X^{**}$ satisfy (1). Then $F|B$ -restriction of F to B is an element of $\ell_0(B)$. For any $b \in B$ $F(b) = \mu_{F|B}(Tb)$ and so, because $sp\, B = X^*$, $F = \tilde{\mu}_{F|B} T \in \varkappa(\mathcal{X})$.

b) Suppose that for any $F \in X^{**}$ satisfying (1) it is $F \in \varkappa(\mathcal{X})$.

Let $y \in \ell_0(B)$. We can extend y linearly continously on the element $F_y \in X^{**}$ satisfying (1). Hence

$$\tilde{\mu}_y T = \tilde{\mu}_{F|B} T = F_y \in \varkappa(\mathcal{X}).$$

Corollary 1. Let X^* have a Mb (B, γ), T is the Dyer embedding of X^* into $c_0(H)$. Then T is w^*-w-continuous iff $\gamma \subset \varkappa(\mathcal{X})$.

Proof. a) Let T be w^*-w-continuous and $F \in \gamma$. Then $\sum_{b \in B} |F(b)| = 1$, so F satisfies (1) and hence $F \in \varkappa(\mathcal{X})$.

b) Let $\gamma \subset \varkappa(\mathcal{X})$. Let $F \in X^{**}$ satisfy (1). Since any finite subset of B is γ-orthogonal, it holds that $\sum_{b \in B} |F(b)| < +\infty$ and so $F = \sum_{b \in B} F(b).F_b$, where $F_b \in \gamma$ so that $F_b(b) = 1$, for any $b \in B$. Hence $F \in \varkappa(\mathcal{X})$.

Corollary 2. Let X^* have a Mb (B, γ) so that $\gamma \subset \varkappa(\mathcal{X})$. Then X is weakly compactly generated.

Proof. The Dyer embedding is then w^*-w-continuous. The closed unit ball in X^* is w^*-compact and hence affinely homeomorphic with some w-compact subset of $c_0(H)$.

From [3], Theorem 3.3 it follows that X is weakly compactly generated.

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Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, Praha 8

Československo

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