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A NOTE ON CARDINAL INVARIANTS OF SQUARE

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Abstract:

This paper contains some results concerning cardinal invariants which appear on $P \times P$, mainly $c(P \times P)$ and $\chi(\Delta)$. Two cases, when the equality $d(P) = c(P \times P)$ holds, are studied and a partition of regular T_1 space into an open dispersed subspace and a closed subspace with prescribed π-weight is given.

Key words and phrases:

Souslin number, density, π-weight, neighbourhood character, linearly ordered topological space, dispersed space.

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The notation of E. Čech, Topological Spaces [1], is used. Cardinal functions are denoted as in Juhász' book [3]. For completeness, the definitions are given here:

Souslin number: $c(P) = \sup \{ \operatorname{card} \mathcal{U} \mid \mathcal{U}$ is a disjoint open system in $P \}$;

density: $d(P) = \min \{ \operatorname{card} \mathcal{D} \mid \mathcal{D}$ is a dense subset of $P \}$;

π-weight: $\pi(P) = \min \{ \operatorname{card} \mathcal{B} \mid \mathcal{B}$ is a π-base for $P \}$;

(A system $\mathcal{B}$ of non-void open subsets of a space P is called π-base for P, if for each open $U \neq \emptyset$ in P there is some $B \in \mathcal{B}$ with $B \subset U$.)

neighbourhood character: $\chi(A \mid P) = \min \{ card\, \mathcal{U} \mid \mathcal{U}$ is a neighbourhood base of a subset A in $P \}$.

$\chi(A \mid P)$ may be abbreviated to $\chi(A)$, if no confusions are possible.

For the other invariants, see [3].

All spaces are assumed to be T_1.

Theorem 1. Let P be a linearly ordered topological space, $n \geq 2$ a natural number. Then $c(P^n) = d(P)$. Particularly, $c(P \times P) = d(P)$.

Proof. Because of the obvious inequality $c(P^n) \leq \leq d(P^n) = d(P)$ we need only to find some dense subset D of P with $card\, D \leq c(P^n)$.

Let $\mathcal{W}$ be the system consisting of all sets of the form $I_1 \times I_2 \times \dots \times I_n$, where $I_1, I_2, \dots, I_n$ are disjoint open intervals in P, and of all singletons $\langle x, x, \dots, x \rangle$, where $x \in P$ is an isolated point. Using Zorn's lemma, one can find a maximal disjoint subsystem $\mathcal{V} \subset \mathcal{W}$. Clearly $card\, \mathcal{V} \leq c(P^n)$.

For $x \in P$, $\langle x, x, \dots, x \rangle \in \overline{\bigcup \mathcal{V}}$: Maximality of $\mathcal{V}$ implies that $\{\langle x, x, \dots, x \rangle\} \in \mathcal{V}$ for every isolated x; suppose x non-isolated, $\langle x, x, \dots, x \rangle \notin \notin \overline{\bigcup \mathcal{V}}$. Then for some open interval $]a, b[$ containing x the cube $]a, b[^n$ is disjoint with $\bigcup \mathcal{V}$. Since x is non-isolated, there must exist a finite sequence $y_1 < y_2 < \dots < y_{n-1}$ of points of $]a, b[$ such that all intervals $]a, y_1[\, ,]y_1, y_2[\, , \dots,]y_{n-2}, y_{n-1}[\, ,]y_{n-1}, b[$ are non-void, but $]a, y_1[\times]y_1, y_2[\times \dots \times]y_{n-1}, b[\in \mathcal{W}$ and $]a, y_1[\times]y_1, y_2[\times \dots \times]y_{n-1}, b[\cap \bigcup \mathcal{V} = \emptyset$, which contradicts to the maximality of $\mathcal{V}$.

Next, put $D = \{x \mid \langle x, x, \dots, x \rangle \in \mathcal{V}\} \cup \{y \mid$ there exists $I_1 \times I_2 \times \dots \times I_n \in \mathcal{V}$ such that y is an end-point of some I_m , $1 \le m \le n \}$. Since $card\, D = card\, \mathcal{V} \le c(P^n)$, it remains to prove that D is dense in P . Pick up a $p \in P$ and let $]u, v[$ be an arbitrary open neighbourhood of p .

We know that $]u, v[^n \cap \bigcup \mathcal{V} \neq \emptyset$, if there exists an $\langle x, x, \dots, x \rangle \in \mathcal{V}$ such that $\langle x, x, \dots, x \rangle \in]u, v[^n$, then $]u, v[\cap D \neq \emptyset$, so let us consider the case $]u, v[^n \cap I_1 \times I_2 \times \dots \times I_n \neq \emptyset$ for some $I_1 \times I_2 \times \dots \times I_n \in \mathcal{V}$ with disjoint $I_1, I_2, \dots, I_n$. Obviously $]u, v[\cap I_j \neq \emptyset$ for all j, $1 \le j \le n$. We claim that at least one end-point of some I_j belongs to $]u, v[$. If not, then $I_j \supset]u, v[$ for every j, $1 \le j \le n$, and since $]u, v[\neq \emptyset$, the intervals $I_1, I_2, \dots, I_n$ cannot be disjoint - a contradiction. Thus $]u, v[$ always meets D and D is dense in P .

<u>**Remark.**</u> **Kurepa's result [4] that for each linearly ordered topological space S the inequality $c(S) \le c(S \times S) \le c(S)^+$ holds, is a consequence of the**

previous theorem. One needs only to realize that the density of a linearly ordered topological space cannot exceed $c(P)^+$. (The proof of this fact, quite adaptable for an arbitrary $c(P)$, is given in Rudin's paper [5] for a special case $c(P) = \aleph_0$.)

The "corner points" of $I_1 \times I_2$ in the proof of Theorem 1 ($m = 2$) have one nice property: they cluster to the diagonal of $P \times P$, as a consequence of linear orderability of the space P. But, without any additional assumptions, the points $x_{U,V}$ chosen arbitrarily from $\overline{U \times V}$, U, V disjoint members of some open base for P, need not behave so nicely and one has to seek them in $W \cap U \times V$, where W is a neighbourhood of the diagonal. This idea leads to the inequality $d(P) \leq \leq \chi(\Delta) \cdot c(P \times P)$, which will appear also as a corollary of the following theorem.

<u>Theorem 2</u>. For a regular space P, $\pi(P) \leq \leq c(P) \cdot \chi(\Delta)$.

<u>Proof</u>: Let $\mathcal{V}$ be a neighbourhood base for Δ in $P \times P$, $card\ \mathcal{V} \leq \chi(\Delta)$. For $V \in \mathcal{V}$ let $\mathfrak{X}_V$ be a system of all non-void open subsets $U \subset P$ such that $U \times U \subset V$. Let $\mathcal{T}_V \subset \mathfrak{X}_V$ be a maximal disjoint subsystem of $\mathfrak{X}_V$ - its existence follows by Zorn's lemma. Since $card\ \mathcal{T}_V \leq c(P)$, for $\mathcal{T} = = \bigcup \{\mathcal{T}_V \mid V \in \mathcal{V}\}$ we have $card\ \mathcal{T} \leq c(P) \cdot \chi(\Delta)$. The desired inequality will follow, if we show that $\mathcal{T}$ is

a π -base.

Let $\mathcal{U}$ be an arbitrary non-void open subset of P ; P being regular, we can find another non-void open subset $\mathcal{U}_1$ such that $\mathcal{U}_1 \subset \overline{\mathcal{U}}_1 \subset \mathcal{U}$. The set $W = (\mathcal{U} \times \mathcal{U}) \cup ((P - \overline{\mathcal{U}}_1) \times (P - \overline{\mathcal{U}}_1))$ is an open neighbourhood of the diagonal; let V be a member of $\mathcal{V}$, $V \subset W$, and consider $\mathcal{T}_V$.

$\bigcup \mathcal{T}_V$ is dense in P because of maximality of $\mathcal{T}_V$. Thus for some $T \in \mathcal{T}_V$ we have $T \cap \mathcal{U}_1 \neq \emptyset$, it contains, say, a point y . By the definition of $\mathfrak{Z}_V$, $T \times T \subset V$. Moreover, $T \subset \mathcal{U}$, which implies that $\mathcal{T}$ is a π -base. To this end, suppose contrary: there exists a point $x \in T - \mathcal{U}$. Then $\langle x, y \rangle \notin \mathcal{U} \times \mathcal{U}$, because $x \notin \mathcal{U}$, $\langle x, y \rangle \notin (P - \overline{\mathcal{U}}_1) \times (P - \overline{\mathcal{U}}_1)$, because $y \in \mathcal{U}_1$, which is a contradiction to $\langle x, y \rangle \in$ $\in T \times T \subset V \subset W = (\mathcal{U} \times \mathcal{U}) \cup ((P - \overline{\mathcal{U}}_1) \times (P - \overline{\mathcal{U}}_1))$.

Remark. Juhász [3] has proved for completely regular spaces P that $w(P) \leq c(P) . u(P)$. The formula given in Theorem 2 is analogous and I do not know whether it can be strengthened to $w(P) \leq c(P) . \psi(\Delta)$.

Corollary 1. For a regular space P

a) $d(P) \leq \pi(P) \leq c(P \times P) . \chi(\Delta)$,

b) $\chi(\Delta) < \pi(P) \Longrightarrow c(P) = d(P) = \pi(P) = c(P \times P)$.

A natural question arises: What are the spaces with neighbourhood character of diagonal less than π -weight like? According to Corollary 1, $\chi(\Delta) < \pi(P)$ holds if and only if $\chi(\Delta) < d(P)$. One consequence of this sharp inequality follows from Theorem 3.

<u>Theorem 3</u>. Let P be a regular space without isolated points. Then $\pi(P) \leq \chi(\Delta)$.

<u>Proof</u>: According to Corollary 1, it suffices to prove the following: Let α be a cardinal number. Then $\chi(\Delta) \leq \alpha$ implies $d(P) \leq \alpha$. The proof will be given in two steps.

I. At first we shall show that under the assumptions of this theorem, each subset of cardinality at least α has a cluster point.

Suppose contrary. There exists an $M \subset P$, $card\ M \geq \geq \alpha$ such that every $x \in P$ has a neighbourhood O_x with $card(O_x \cap M) \leq 1$. Without loss of generality we may assume that $card\ M = \chi(\Delta)$.

Let $\mathcal{U}$ be a neighbourhood base of Δ, $card\ \mathcal{U} = = \chi(\Delta)$. The cardinality of $\mathcal{U}$ equals to that of M , hence we may write $\mathcal{U} = \{U_x \mid x \in M\}$. Since P has no isolated point, no $x \in M$ is isolated and thus for each $U \times \mathcal{U}$ there exists an $y_x \neq x$ such that $\langle x, y_x \rangle \in U_x$.

Clearly $cl\{\langle x, y_x\rangle \mid x \in M\} \cap \Delta = \emptyset$ - if not, one obtains a contradiction to discreteness of M . Thus $V = P \times P - cl\{\langle x, y_x \rangle \mid x \in M\}$ is an open subset of $P \times P$ containing the diagonal; since $\mathcal{U}$ is a neighbourhood base of Δ , there is some $U_x \in \mathcal{U}$, $U_. \subset V$. But $\langle x, y_x\rangle \in U_x$, $\langle x, y_x \rangle \notin V$

- a contradiction.

II. Now we shall construct a dense set in P of cardinality $\leq \alpha$.

Again, let $\mathcal{U}$ be a neighbourhood base of Δ , $card\, \mathcal{U} \leq \alpha$. For each $U \in \mathcal{U}$ there exists a subset $A_U \subset P$ such that

(i) $x \neq y,\ x, y \in A_U \Rightarrow \langle x, y \rangle \notin U$,

(ii) $A' \supsetneqq A_U \Rightarrow \exists x, y \in A',\ x \neq y,\ \langle x, y \rangle \in U$.

(In the system $\mathcal{A}$ of all $A \subset P$ satisfying (i), define a partial order by inclusion. Then apply Zorn's lemma and denote any maximal element by A_U . It will satisfy (ii), too.)

A_U is discrete (in P) for every U . Suppose contrary: Let an $x \in P$ be a cluster point of A_U . For every open neighbourhood O of x we have $card\,(O \cap A_U) \geq \aleph_0$; since U is a neighbourhood of Δ , there is a neighbourhood O_x of x with $O_x \times O_x \subset U$. Let us pick up two distinct points y, z belonging to $A_U \cap O_x$. Then $\langle y, z \rangle \in O_x \times O_x \subset U$, which is a contradiction to (i). Following I, we obtain $card\, A_U < \alpha$.

Let us denote $A = \bigcup \{ A_U \mid U \in \mathcal{U} \}$. Obviously $card\, A \leq \alpha$. The set A is dense in P : For any $x \in P$, $x \notin A$, let us choose an open neighbourhood O of x and (P regular) let us find some open V with $x \in V \subset \overline{V} \subset O$. The set $W = (O \times O) \cup ((P - \overline{V}) \times (P - \overline{V}))$ is a neighbourhood of Δ in $P \times P$, hence there is some $\mathcal{U} \in \mathfrak{U}$ contained in W. It remains to show that O intersects $A_{\mathcal{U}}$. Setting $A' = A_{\mathcal{U}} \cup \cup \{x\}$, there must be some y in $A_{\mathcal{U}}$ with $\langle x, y \rangle \in \in \mathcal{U}$ by (ii). Since $\mathcal{U} \subset (O \times O) \cup ((P - \overline{V}) \times (P - \overline{V}))$, the point $\langle x, y \rangle$ belongs to $O \times O$ and the point y belongs to $O \cap A_{\mathcal{U}}$. This completes the proof.

Corollary 2. Let P be regular, $\chi(\Delta) < \pi(P)$. Then P contains at least one isolated point.

Lemma. Let P be a topological space, A a closed subset of P. Then $\chi(\Delta_A \mid A \times A) \leq \chi(\Delta_P \mid P \times P)$.

The proof is easy and may be left to the reader.

Corollary 3. Let P be regular. Then $P = A \cup B$, where $A \cap B = \emptyset$, A is closed in P, $\pi(A) \leq \leq \chi(\Delta_P \mid P \times P)$ and B is dispersed. If $\chi(\Delta_P \mid P \times P) < \pi(P)$, then $\operatorname{card} B \geq \pi(P)$.

Proof: If $\pi(P) \leq \chi(\Delta_P \mid P \times P)$, it suffices to write $A = P$, $B = \emptyset$. If $\pi(P) > \chi(\Delta_P \mid P \times P)$, there are isolated points in P by Corollary 2. The reader may verify that the cardinality of the set of isolated points is greater or equal to $\pi(P)$.

Let us define for ordinal numbers ξ, $\operatorname{card} \xi < < \operatorname{card} P$, the sets A_ξ, B_ξ, C_ξ:

$$\xi = 0 : C_0 = B_0 = \{x \in P \mid x \text{ isolated in } P\}$$

$$A_0 = P - B_0 ;$$

$\xi = \beta + 1$: $C_\xi = \{x \mid x \in A_\beta, x$ isolated in $A_\beta\}$

$$B_\xi = \bigcup \{B_\alpha \mid \alpha < \xi\} \cup C_\xi$$

$$A_\xi = P - B_\xi \ ;$$

ξ limit ordinal: $C_\xi = \emptyset$, $B_\xi = \bigcup \{B_\alpha \mid \alpha < \xi\}$,

$$A_\xi = P - B_\xi \ .$$

Obviously A_ξ is closed for every ξ , thus, by the Lemma, $\chi(\Delta_{A_\xi} \mid A_\xi \times A_\xi) \leq \chi(\Delta_P \mid P \times P)$.

Let η be the first ordinal such that $\chi(\Delta_{A_\eta} \mid A_\eta \times A_\eta) \geq \pi(A_\eta)$. It remains to write $A = A_\eta$, $B = B_\eta$.

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