Commentationes Mathematicae Universitatis Carolinae

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On skew lattices. I.

Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 1, 73--85

Persistent URL: http://dml.cz/dmlcz/105471

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14,1 (1973)

ON SKEW LATTICES I

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Abstract: In this paper a method is given which enables us to transfer some theorems of lattice theory into theorems on skew lattices. The results are applied to the case of distributive and modular lattices.

Key words: Skew lattice, variety of lattices

AMS, Primary: 06A20 Ref. Ž. 2.724.8

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1. Introduction. In the present paper we shall prove the theorems which generalize some results of the theory of lattices, especially the properties of lattices which can be expressed by the lattice theoretical formulas. By an application of these theorems we get a generalization of some results known in the theory of distributive and modular lattices.

Algebras (skew lattices) will be denoted by German capital letters and the base set by the corresponding Latin capital letters. If $\mathfrak{M}$ is a skew lattice and θ a congruence relation on $\mathfrak{M}$, we shall denote the factor skew lattice by

$\mathfrak{M}/\theta$ and the base set by M/θ. Let us fix an infinite countable set and denote it by X; its elements are called variables. Let us denote by $\mathfrak{W}$ an absolutely free algebra of type $(2,2)$ generated by X. The elements of $\mathfrak{W}$ are called terms. By a formula we mean a formula of the language $\{\wedge, \vee\}$ (in variables from X), by a theory an arbitrary set of formulas is meant here. We shall denote the class of all models of a theory T by $Mod(T)$.

2. Definition and basic properties.

2.1. Definition. A skew lattice is an algebra $\mathfrak{M} = \langle M, \wedge, \vee \rangle$ where $\wedge$ and $\vee$ are two binary operations on M, called meet and join respectively, satisfying the following laws for all $a, b, c \in M$:

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c, \quad a \vee (b \vee c) = (a \vee b) \vee c,$$
$$a \wedge (b \vee a) = a, \quad (a \wedge b) \vee a = a,$$
$$a \wedge (a \vee b) = a, \quad (b \wedge a) \vee a = a.$$

2.2. Definition. Let $\mathfrak{M}$ be a skew lattice. We define binary relations $\leq$ and $\equiv$ on M by the following:

(i) $a \leq b$ iff $a \wedge b = a$;

(ii) $a \equiv b$ iff $a \wedge b = a$ and $b \wedge a = b$.

2.3. Theorem. Let $\mathfrak{M}$ be a skew lattice and $a, b, c, d \in M$. Then the following conditions are satisfied:

(i) $a \wedge b = (a \wedge b) \wedge a$, $\quad b \vee a = a \vee (b \vee a)$;

(ii) $a \wedge a = a$, $\quad a \vee a = a$;

(iii) $a \leq b$ iff $a \vee b = b$;

(iv) $a \wedge b \leq a$ and $a \wedge b \leq b$;

(v) $a \leq a \vee b$ and $b \leq a \vee b$;

(vi) $a \leq b$ and $c \leq d$ imply $a \wedge c \leq b \wedge d$;

(vii) $a \leq b$ and $c \leq d$ imply $a \vee c \leq b \vee d$;

(viii) $\leq$ is a quasi-ordering on M ;

(ix) $\equiv$ is a congruence relation on $\mathfrak{M}$.

The proof of 2.3 is not difficult (see [2]).

2.4. <u>Theorem</u>. Let $\mathfrak{M}$ be a skew lattice. Then $\mathfrak{M}/\equiv$ is the modification of $\mathfrak{M}$ in the variety of lattices.

<u>Proof</u>. It is clear that $\mathfrak{M}/\equiv$ is a lattice. Let $\mathfrak{L}$ be a lattice and φ a homomorphism of $\mathfrak{M}$ into $\mathfrak{L}$. We denote the natural homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}/\equiv$ by ν. For each $a = m\nu \in M/\equiv$ we define $a\psi = m\psi$. Obviously, ψ is a homomorphism of $\mathfrak{M}/\equiv$ into $\mathfrak{L}$ such that $\nu\psi = \varphi$.

2.5. <u>Definition</u>. Let $\mathfrak{M}$ be a skew lattice. A subset I of the set M is called an ideal of $\mathfrak{M}$ iff

(i) $a, b \in I$ implies that $a \vee b \in I$;

(ii) $a \in I$ and $b \leq a$ imply $b \in I$.

2.6. <u>Theorem</u>. The set of all ideals of a skew lattice forms a complete lattice (with respect to the set-inclusion) which is isomorphic to the lattice of all ideals of the lattice $\mathfrak{M}/\equiv$.

<u>Proof</u>. The first part of the theorem is trivial. Let us denote ν the natural homomorphism of $\mathfrak{M}$ onto $\mathfrak{M}/\equiv$.

It is easy to verify that a subset I of the set M is an ideal of $\mathfrak{M}$ if and only if the set $I\nu = \{i\nu ; i \in I\}$ is an ideal of the lattice $\mathfrak{M}/\equiv$. If K, L are ideals of $\mathfrak{M}$, then $K \subseteq L$ is equivalent to $K\nu \subseteq L\nu$ and if J is an ideal of $\mathfrak{M}/\equiv$ then the set $J\nu^{-1} = \{a \in \in M; a\nu \in J\}$ is an ideal of $\mathfrak{M}$ such that $(J\nu^{-1})\nu = J$. So we get that the mapping $I \longmapsto I\nu$ is a complete isomorphism of the lattice of all ideals of $\mathfrak{M}$ onto the lattice of all ideals of $\mathfrak{M}/\equiv$.

Duality Principle. The dual term to a term t is defined by the following two rules:

1) For all variables x, $D(x) = x$.

2) If t_1, t_2 are terms, then $D(t_1 \wedge t_2) = D(t_2) \vee D(t_1)$ and $D(t_1 \vee t_2) = D(t_2) \wedge D(t_1)$.

For an arbitrary formula let $D(\varphi)$ denote the formula obtained from φ in such a way that each term occurring in φ is replaced by its dual term. The formula $D(\varphi)$ is said to be dual to φ . The dual theory $D(T)$ of a theory T is defined as the set of all $D(\varphi)$ where φ is an element of T . A theory T is said to be self-dual iff $D(T) = T$.

We shall denote the theory of skew lattices (i.e. the set of its axioms) and the theory of lattices by T_{SL} and T_L respectively. It is clear that the theory T_{SL} is self-dual and so we have

2.7. <u>Theorem</u>. Let T be a self-dual theory. Then a formula φ is a consequence of the theory $T_{SL} \cup T$ if and only if the formula $\mathcal{D}(\varphi)$ is a consequence of $T_{SL} \cup T$.

Let $\mathfrak{M} = \langle M, \wedge, \vee \rangle$ be a skew lattice. If we define the operations $\cap, \cup$ on M by

$$a \cap b = b \vee a \qquad a \cup b = b \wedge a ,$$

then the algebra $\mathcal{D}(\mathfrak{M}) = \langle M, \cap, \cup \rangle$ is again a skew lattice and it is said to be the dual skew lattice of $\mathfrak{M}$.

3. <u>Main results</u>. Let φ be a formula. The formula obtained from φ in such a way that each equation $p = q$ occurring in φ is replaced by the formula $p \wedge q = p$ & & $q \wedge p = q$ will be denoted by φ^*. For a theory T, we denote the set of all formulas φ^* where $\varphi \in T$ by T^*. The natural homomorphism of a skew lattice $\mathfrak{M}$ onto $\mathfrak{M}/\equiv$ will be denoted by ν_M. We shall suppose all classes of lattices used below closed under isomorphic images. If K is

a class of lattices, then the class of all skew lattices $\mathfrak{M}$ with $\mathfrak{M}/\equiv \in K$ will be denoted by $\mathcal{S}(K)$.

3.1. Lemma. Let $\mathfrak{M}$ be a skew lattice, let p, q be terms and let α be a mapping of X into M. Let $\bar{\alpha}$ denote the homomorphism of $\mathfrak{W}$ into $\mathfrak{M}$ extending the mapping α. Then the following statements are equivalent:

1) The formula $(p = q)^*$ is satisfied by α in $\mathfrak{M}$.

2) $p\bar{\alpha} \equiv q\bar{\alpha}$.

3) The formula $p = q$ is satisfied by $\alpha\nu_M$ in $\mathfrak{M}/\equiv$.

3.2. Proposition. Let $\mathfrak{M}$ be a skew lattice and let φ be a formula. Then the formula φ^* is satisfied by $\alpha : X \to M$ in $\mathfrak{M}$ if and only if the formula φ is satisfied by $\alpha\nu_M$ in $\mathfrak{M}/\equiv$.

Proof. Let Γ denote the set of all formulas φ having the property that φ^* is satisfied by α in $\mathfrak{M}$ if and only if the formula φ is satisfied by $\alpha\nu_M$ in $\mathfrak{M}/\equiv$. By 3.1 Γ contains all equations. It is clear that $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$ imply $\varphi_1 \vee \varphi_2$, $\neg\varphi_1$ belong to Γ. We shall prove that $\varphi \in \Gamma$ implies $(\exists x)\varphi \in \Gamma$. Let $(\exists x)\varphi^*$ be satisfied by α in $\mathfrak{M}$. Then there exists $\beta : X \to M$

such that φ^* is satisfied by β in $\mathfrak{M}$ and $\beta|X-\{x\} = \alpha|X-\{x\}$. The formula $\varphi \in \Gamma$ and, thus, φ is satisfied by $\beta\nu_M$ in $\mathfrak{M}/\equiv$. Suppose that the formula $(\exists x)\varphi$ is satisfied by $\alpha\nu_M$ in $\mathfrak{M}/\equiv$. Then there exists $\gamma : X \longrightarrow M/\equiv$ such that the formula φ is satisfied by γ in $\mathfrak{M}$ and $\gamma|X-\{x\} = \alpha|X-\{x\}$ and $\beta\nu_M = \gamma$.

So we get that φ^* is satisfied by β in $\mathfrak{M}$ and, hence, we can see that $(\exists x)\varphi^*$ is satisfied by α in $\mathfrak{M}$. Thus, Γ is the set of all formulas.

Since every mapping of X into $M/\equiv$ can be represented as a product of a mapping of X into M and of the mapping ν_M, we have the following result:

3.3. <u>Theorem</u>. Let $\mathfrak{M}$ be a skew lattice and let φ be a formula. Then the formula φ^* is satisfied in $\mathfrak{M}$ if and only if the formula φ is satisfied in $\mathfrak{M}/\equiv$.

3.4. <u>Corollary</u>. A formula φ is satisfied in a lattice $\mathcal{L}$ if and only if the formula φ^* is satisfied in $\mathcal{L}$.

3.5. <u>Corollary</u>. Let $\mathfrak{M}$ be a skew lattice and let p, q be terms. Then the following statements are equivalent:

(i) The equations $p \wedge q = p$, $q \wedge p = q$ are satisfied in $\mathfrak{M}$.

(2) For each homomorphism ϑ of $\mathfrak{P}$ into $\mathfrak{M}$

$p\vartheta \equiv q\vartheta$.

(3) The equation $p = q$ is satisfied in $\mathfrak{M}/\equiv$.

3.6. Lemma. Let K be a class of lattices and let φ be a formula. The following two statements are equivalent:

(1) If φ is satisfied in a lattice $\mathcal{L}$, then $\mathcal{L} \in K$.

(2) If φ^* is satisfied in a skew lattice $\mathfrak{M}$, then $\mathfrak{M} \in \mathcal{S}(K)$.

3.7. Lemma. Let K be a class of lattices and let φ be a formula. The following two statements are equivalent:

(1) If $\mathcal{L} \in K$, then φ is satisfied in $\mathcal{L}$.

(2) If $\mathfrak{M} \in \mathcal{S}(K)$, then φ^* is satisfied in $\mathfrak{M}$.

The proofs of 3.6 and 3.7 are straightforward, using 3.3 and 3.4.

3.8. Theorem. A class K of lattices is axiomatic (elementary) if and only if the class $\mathcal{S}(K)$ is axiomatic (elementary). Moreover, if $K = Mod(T_L \cup T)$ where T is an arbitrary theory, then $\mathcal{S}(K) = Mod(T_{SL} \cup T^*)$.

3.9. Theorem. Let T_1, T_2 be theories. The following statements are equivalent:

(1) $Mod(T_L \cup T_1) \subseteq Mod(T_L \cup T_2)$.

(2) $Mod(T_{SL} \cup T_1^*) \subseteq Mod(T_{SL} \cup T_2^*)$.

The proofs of 3.8 and 3.9 can be deduced immediately from 3.6 and 3.7.

Note. The inclusion in 3.9 can be replaced by the equality.

3.10. Theorem. Let $\mathcal{K}$ be a variety (quasi-variety) of lattices. Then $\mathcal{S}(\mathcal{K})$ is a variety (quasi-variety) of skew lattices.

Proof. We can assume that $\mathcal{K} = Mod(T_L \cup T)$ where T is a set of equations (quasi-equations). By 3.8 $\mathcal{S}(\mathcal{K}) = Mod(T_{SL} \cup T^*)$. Let T^0 denote the theory obtained from T by replacing each formula

$\varphi^* = \varphi_1 \,\&\, \varphi_2 \,((\varphi_1^* \,\&\, \dots \,\&\, \varphi_K^* \rightarrow \psi^*) = (\varphi_1^* \,\&\, \dots \,\&\, \varphi_K^* \rightarrow \psi_1 \,\&\, \psi_2))$

from T by two equations (quasi-equations)

$\varphi_1, \varphi_2 \,(\varphi_1^* \,\&\, \dots \,\&\, \varphi_K^* \rightarrow \psi_1 ,\ \varphi_1^* \,\&\, \dots \,\&\, \varphi_K^* \rightarrow \psi_2)$.

Thus we get a set T^0 of equations (quasi-equations) such that $Mod(T_{SL} \cup T^0) = Mod(T_{SL} \cup T^*)$. Hence $\mathcal{S}(\mathcal{K})$ is a variety (quasi-variety) of skew lattices.

3.11. Theorem. Let $\mathcal{K}$ be a class of lattices. The following two statements are equivalent:

(1) A lattice $\mathcal{L} \in \mathcal{K}$ if and only if the lattice of all ideals of $\mathcal{L}$ belongs to $\mathcal{K}$.

(2) A skew lattice $\mathfrak{M} \in \mathcal{S}(\mathcal{K})$ if and only if the lattice of all ideals of $\mathfrak{M}$ belongs to $\mathcal{K}$.

Proof. By 2.6 the lattice of all ideals of $\mathfrak{M}$ is isomorphic to the lattice of all ideals of $\mathfrak{M}/\equiv$ for every skew lattice $\mathfrak{M}$. From this fact the theorem follows immediately.

It is easy to show that the lattices $D(\mathfrak{M}/\equiv)$ and $D(\mathfrak{M})/\equiv$ are isomorphic for every skew lattice $\mathfrak{M}$. Thus we have

3.12. Theorem. Let K be a class of lattices. Then the class K contains with a lattice $\mathcal{L}$ its dual $D(\mathcal{L})$ if and only if the class $\mathcal{S}(K)$ contains with a skew lattice $\mathfrak{M}$ its dual $D(\mathfrak{M})$.

4. Weak distributive and modular skew lattices. The results obtained in the previous chapter will now be applied to the case of distributive and modular lattices. In this way we shall obtain generalizations of some results concerning these lattices.

4.1. Definition. A skew lattice $\mathfrak{M}$ is called weak distributive iff for all $a, b, c \in M$ $a \wedge (b \vee c) \equiv$ $\equiv (a \wedge b) \vee (a \wedge c)$.

4.2. Remark. A skew lattice $\mathfrak{M}$ is weak distributive if and only if for each homomorphism ϑ of $\mathfrak{W}$ into $\mathfrak{M}$ $p\vartheta \equiv q\vartheta$ holds where $p = x_1 \wedge (x_2 \vee x_3)$ and $q =$ $= (x_1 \wedge x_2) \vee (x_1 \wedge x_3)$. By 3.5 we get that a skew lattice $\mathfrak{M}$ is weak distributive if and only if the equations $p \wedge q = p$ and $q \wedge p = q$ are satisfied in $\mathfrak{M}$. Since the equation $q \wedge p = q$ is satisfied in every skew lattice, we have that a skew lattice is weak distributive if and only if for all $a, b, c \in M$

$(a \wedge (b \vee c)) \wedge ((a \wedge b) \vee (a \wedge c)) = a \wedge (b \vee c)$.

Thus, the class of all weak distributive skew lattices is equational and it can be characterized by one equation. From 3.5 it also follows that a skew lattice $\mathfrak{M}$ is weak distributive if and only if the lattice $\mathfrak{M}/\equiv$ is distributive. If we denote the class of all distributive lattices by K_D , then the class of all weak distributive skew lattices is equal to $\mathcal{S}(K_D)$.

4.3. Theorem. Let $\mathfrak{M}$ be a skew lattice. The following conditions are equivalent:

(1) $\mathfrak{M}$ is weak distributive.

(2) For all $a, b, c \in M$ $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$.

(3) For all $a, b, c \in M$ $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \equiv$
$\equiv (a \vee b) \wedge (a \vee c) \wedge (b \vee c)$.

(4) If $a, b, c \in M$ are such that $a \wedge b \equiv a \wedge c$ and $a \vee b \equiv$
$\equiv a \vee c$, then $b \equiv c$.

(5) For all $a, b, c \in M$ $(a \vee b) \wedge (a \vee c) \wedge (a \vee (b \wedge c)) =$
$= (a \vee b) \wedge (a \vee c)$.

(6) $D(\mathfrak{M})$ is distributive.

(7) The lattice of all ideals of $\mathfrak{M}$ is distributive.

Proof. The equivalence of the conditions (1),(2),(3), (4),(5) can be deduced from 3.9. The equivalence of the conditions (1,(6) and the one of (1),(7) is a trivial consequence of 3.11 and 3.12 respectively.

4.4. Definition. A skew lattice $\mathfrak{M}$ is called weak modular iff for all $a, b, c \in M$

$a \vee (b \wedge (a \vee c)) \equiv (a \vee b) \wedge (a \vee c)$.

4.5. Remark. By considerations similar to the ones in 4.2 we can get that the class of all weak modular skew lattices is equal to $\mathcal{S}(K_M)$ where K_M denotes the class of all modular lattices and it can be characterized by the following equation:

$$(x_1 \vee x_2) \wedge (x_1 \vee x_3) \wedge (x_1 \vee (x_2 \wedge (x_1 \vee x_3))) = (x_1 \vee x_2) \wedge (x_1 \vee x_3).$$

4.6. Theorem. Let $\mathfrak{M}$ be a skew lattice. The following conditions are equivalent:

(1) $\mathfrak{M}$ is weak modular.

(2) For all $a, b, c \in M$ $a \wedge (b \vee (a \wedge c)) \equiv (a \wedge b) \vee (a \wedge c)$.

(3) If $a, b, c \in M$ are such that $a \leq b$, $a \wedge c \equiv b \wedge c$ and $a \vee c \equiv b \vee c$, then $a \equiv b$.

(4) If $a, b, c \in M$ are such that $a \leq c$, then

$(a \vee b) \wedge (a \vee c) \wedge (b \vee c) \equiv (a \wedge b) \vee (a \wedge c) \vee (b \wedge c)$.

(5) $D(\mathfrak{M})$ is weak modular.

(6) The lattice of all ideals of $\mathfrak{M}$ is modular.

The proof of 4.6 is similar to the one of 4.3.

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(Oblatum 6.12.1972)