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AFFINE H-STRUCTURES

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1. Introduction. In [1] Lüneburg defines affine Hjelmslev planes as incidence structures (without the claim of uniqueness of join and intersection) with parallelism and relation of neighbouring points, respectively, neighbouring lines. He has found many deep results about such planes.

Our aim is a suitable generalization of some considerations in [1]. A notion of an affine Hjelmslev structure is being defined and its first properties are studied here.

2. Definitions and some properties of affine H-structures.

Definition 1. Let T be a (multiplicative) group and $\mathfrak{B}$ some system of its non-trivial subgroups. We define a geometric structure $(T, \mathfrak{B})$ with the following properties:

(i) points of $(T, \mathfrak{B})$ are defined as elements in T,

(ii) lines of $(T, \mathfrak{B})$ are defined as right cosets with respect to the subgroups in $\mathfrak{B}$,

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(iii) a point $\mathfrak{p}$ is called incidental with a line $P\mathfrak{q}$, when $\mathfrak{p} \in P\mathfrak{q}$.

(iv) lines $P\mathfrak{q}$, $Q\mathfrak{p}$ will be called parallel, when $P = Q$ (see [1], [2]).

Remark. We see instantly, that $(T, \mathfrak{B})$ is an incidence structure with parallelism (in the sense of [1]) such that (*): Given a point $\mathfrak{p}$ and a line $Q\mathfrak{r}$ not through $\mathfrak{p}$ there exists a unique line through $\mathfrak{p}$ parallel to $Q\mathfrak{r}$.

Definition 2. By a partition of a group T it is meant a system $\mathfrak{B}$ of non-trivial [1] subgroups of T such that

1) $\mathfrak{B}$ covers T,

2) $P, Q \in \mathfrak{B}, P \neq Q \Longrightarrow |P \cap Q| = 1$ [2], (see [2]).

Definition 3. A geometric structure $(T, \mathfrak{B})$ is called an affine Hjelmslev structure (H-structure), if it satisfies:

H 1. $\mathfrak{B}$ is a covering of T.

H 2. There exists an epimorphism φ of T on a suitable group $\overline{T}$ such that

a) $\overline{\mathfrak{B}} = \{P^{\varphi} \mid P \in \mathfrak{B}\}$ is a partition of $\overline{T}$,

b) $|P \cap Q| = 1 \Longleftrightarrow P^{\varphi} \neq Q^{\varphi}$, $\forall P, Q \in \mathfrak{B}$;

H 3. $\operatorname{Ker} \varphi = N = \bigcup_{\{P,Q\} \subseteq \mathfrak{B}} (P \cap Q)$.

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1) trivial = of order 1, non-trivial = of order > 1.

2) $|\ |$ = the cardinality.

Theorem. A geometric structure $(T, \mathcal{B})$ is an affine H-structure if and only if

α) $\mathcal{B}$ is a covering of T,

β) $N = \bigcup_{\{P, Q\} \subseteq \mathcal{B}} (P \cap Q)$, for $P \neq Q$, is a normal subgroup of T,

γ) $|P \cap Q| = 1 \Rightarrow N = PN \cap QN$, where P, $Q \in \mathcal{B}$,

δ) $|P \cap Q| \neq 1 \Rightarrow PN = QN$, where $P, Q \in \mathcal{B}$.

Proof. Let $(T, \mathcal{B})$ be a geometric structure satisfying $\alpha) - \delta)$. H 1 is satisfied trivially. In order to prove the validity of H 2, let $\varphi : T \to \bar{T} = T/N$ be the canonical epimorphism. $\bar{\mathcal{B}} = \{P^{\varphi} \mid P \in \mathcal{B}\}$ consists of subgroups of $\bar{T}$ and covers $\bar{T}$, because of α). Now suppose $|P^{\varphi} \cap Q^{\varphi}| \neq 1$. Then $PN \cap QN \neq N$ and $|P \cap Q| \neq 1$ according to γ). By virtue of δ) $PN = QN$. Hence $P^{\varphi} = Q^{\varphi}$ and $|P^{\varphi} \cap Q^{\varphi}| \neq 1$. Finally β) implies H 3.

Conversely, suppose that $(T, \mathcal{B})$ is an affine H-structure. Then H 1 implies α), H 3 implies β), H 2b implies γ) and finally H 2 a)b) implies δ).

Corollary 1. If in an affine H-structure $(T, \mathcal{B})$ evèry two distinct subgroups of $\mathcal{B}$ have at least two elements in common, then $|\bar{\mathcal{B}}| = 1$.

The proof follows at once from Definition 3 and the preceding theorem.

Corollary 2. Let $(T, \mathcal{B})$ be an affine H-structure. Let R be a normal subgroup of T and $R \subseteq N$ (N

is defined in Theorem). Then the geometric structure $(T', \mathfrak{B}') = (T/R, \{\{R p \mid p \in P\} \mid P \in \mathfrak{B}\}$ is an affine H-structure.

To the proof: It is only a routine to verify the validity of the corresponding conditions α) to σ).

3. Example. Let there be an additive group of the 6-dimensional vector space over the two-element field. Consider the vectors as 6-tuple rows and subgroups in T as matrices with six columns, then the following system of subgroups of T is investigated.

$$\mathfrak{B} = \Bigg\{ {}^1P_1 = \begin{pmatrix} 000000 \\ 000111 \\ 111111 \\ 111000 \end{pmatrix}, \quad {}^1P_2 = \begin{pmatrix} 000000 \\ 000111 \\ 111110 \\ 111001 \end{pmatrix}, \quad {}^1P_3 = \begin{pmatrix} 000000 \\ 000111 \\ 111101 \\ 111010 \end{pmatrix},$$

$${}^1P_4 = \begin{pmatrix} 000000 \\ 000111 \\ 111011 \\ 111100 \end{pmatrix}, \quad {}^2P_1 = \begin{pmatrix} 000000 \\ 000110 \\ 110000 \\ 110110 \end{pmatrix}, \quad {}^2P_2 = \begin{pmatrix} 000000 \\ 000110 \\ 110111 \\ 110001 \end{pmatrix},$$

$${}^2P_3 = \begin{pmatrix} 000000 \\ 000110 \\ 110101 \\ 110011 \end{pmatrix}, \quad {}^2P_4 = \begin{pmatrix} 000000 \\ 000110 \\ 110100 \\ 110010 \end{pmatrix}, \quad {}^3P_1 = \begin{pmatrix} 000000 \\ 000101 \\ 101000 \\ 101101 \end{pmatrix},$$

$${}^3P_2 = \begin{pmatrix} 000000 \\ 000101 \\ 101111 \\ 101010 \end{pmatrix}, \quad {}^3P_3 = \begin{pmatrix} 000000 \\ 000101 \\ 101100 \\ 101001 \end{pmatrix}, \quad {}^3P_4 = \begin{pmatrix} 000000 \\ 000101 \\ 101110 \\ 101011 \end{pmatrix},$$

$${}^4P_1 = \begin{pmatrix} 000000 \\ 000011 \\ 011011 \\ 011000 \end{pmatrix}, \quad {}^4P_2 = \begin{pmatrix} 000000 \\ 000011 \\ 011111 \\ 011100 \end{pmatrix}, \quad {}^4P_3 = \begin{pmatrix} 000000 \\ 000011 \\ 011110 \\ 011101 \end{pmatrix},$$

$${}^4P_4 = \begin{pmatrix} 000000 \\ 000011 \\ 011001 \\ 011010 \end{pmatrix}, \quad {}^5P_1 = \begin{pmatrix} 000000 \\ 000100 \\ 100100 \\ 100000 \end{pmatrix}, \quad {}^5P_2 = \begin{pmatrix} 000000 \\ 000100 \\ 100111 \\ 100011 \end{pmatrix},$$

$${}^5P_3 = \begin{pmatrix} 000000 \\ 000100 \\ 100010 \\ 100110 \end{pmatrix}, \quad {}^5P_4 = \begin{pmatrix} 000000 \\ 000100 \\ 100001 \\ 100101 \end{pmatrix}, \quad {}^6P_1 = \begin{pmatrix} 000000 \\ 000010 \\ 010000 \\ 010010 \end{pmatrix},$$

$$
{}^6P_2 = \begin{pmatrix} 000000 \\ 000010 \\ 010111 \\ 010101 \end{pmatrix}, \quad {}^6P_3 = \begin{pmatrix} 000000 \\ 000010 \\ 010110 \\ 010100 \end{pmatrix}, \quad {}^6P_4 = \begin{pmatrix} 000000 \\ 000010 \\ 010011 \\ 010001 \end{pmatrix},
$$

$$
{}^7P_1 = \begin{pmatrix} 000000 \\ 000001 \\ 001000 \\ 001001 \end{pmatrix}, \quad {}^7P_2 = \begin{pmatrix} 000000 \\ 000001 \\ 001111 \\ 001110 \end{pmatrix}, \quad {}^7P_3 = \begin{pmatrix} 000000 \\ 000001 \\ 001100 \\ 001101 \end{pmatrix},
$$

$$
{}^7P_4 = \begin{pmatrix} 000000 \\ 000001 \\ 001010 \\ 001011 \end{pmatrix} \Bigg\} .
$$

Evidently $\mathcal{B}$ is a covering of T. For N, by its definition in Theorem, we obtain

$$
\begin{pmatrix} 000000 \\ 000100 \\ 000010 \\ 000001 \\ 000110 \\ 000101 \\ 000011 \\ 000111 \end{pmatrix}
$$

for every distinct ${}^iP_j$, ${}^iP_k$, $i = 1,2,3,4,5,6,7$; $j,k = 1,2,3,4$; we verify $|{}^iP_j \cap {}^iP_k| = 2$ so that ${}^iP_j + N = {}^iP_k + N$. Further for every distinct ${}^iP_j$, ${}^kP_m$, where $i,k = 1,2,3,4,5,6,7$; $j,m = 1,2,3,4$, we obtain $|{}^iP_j \cap {}^kP_m| = 1$ from which $({}^iP_j + N) \cap ({}^kP_m + N) = N$. In this way we are able to verify the properties α) to δ), so that by Theorem $(T, \mathcal{B})$ becomes an affine H-structure. We shall still note what is the meaning of $\overline{T}$ and φ from Definition 3 in our example: $\overline{T}$ is the 3-dimensional vector space over the two-element field and the epimorphism $\varphi : T \to \overline{T}$ is given by $(a_1\, a_2\, a_3\, a_4\, a_5\, a_6)\varphi = (a_1\, a_2\, a_3)$, $a_i = 0, 1$.

Furthermore, it must hold $\mathrm{Ker}\,\varphi = \mathcal{N}$ and

$$\bar{\mathfrak{B}} = \left\{ \begin{pmatrix} 000 \\ 111 \end{pmatrix}, \begin{pmatrix} 000 \\ 110 \end{pmatrix}, \begin{pmatrix} 000 \\ 101 \end{pmatrix}, \begin{pmatrix} 000 \\ 011 \end{pmatrix}, \begin{pmatrix} 000 \\ 100 \end{pmatrix}, \begin{pmatrix} 000 \\ 010 \end{pmatrix}, \begin{pmatrix} 000 \\ 001 \end{pmatrix} \right\}$$

must be a partition of $\bar{T}$. As the final remark we describe the geometric interpretation of our example. The mapping φ is nothing else as the orthogonal projection of V^6 onto V^3. The constructed partition of V^3 is the set of all one-dimensional subspaces in V^3 and $\mathrm{Ker}\,\varphi$ is a vector space W^3, which results to be the complement of V^3. The system of subgroups of $\mathfrak{B}$, forming the desired covering of V^6, is such that 1) the preimage of every 1-dimensional member of $\mathfrak{B}$ is decomposed onto 2-dimensional subspaces having a 1-dimensional subspace of W^3 as their common intersection and 2) every two 2-dimensional members of $\mathfrak{B}$ belonging to distinct preimages are disjoint.

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