Commentationes Mathematicae Universitatis Carolinae

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Commentationes Mathematicae Universitatis Carolinae, Vol. 13 (1972), No. 4, 617--627

Persistent URL: http://dml.cz/dmlcz/105446

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13,4 (1972)

FIXED POINT THEOREMS FOR GENERALIZED CONTRACTIONS

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W.V. Petryshyn has given in [7] some fixed point theorems on so called [3],[4] "generalized contractions" (Def. 1 (i)) and on "uniformly generalized contractions" (Def. 1 (ii)) proving them by a degree argument (and therefore function's domains must have interior points). We strengthen and generalize some of these results by a unifying and elementary approach, using methods discussed in [3],[4],[5],[8],[9].

Definition 1: Let $(E, \| \ \|)$ be a normed linear space and $\emptyset \neq X \subset E$;

(i) $f: X \longrightarrow E$ is said to be a "generalized contraction": $\Longleftrightarrow$

$$(*) \quad \bigvee_{\alpha: X \to [0,1)} \ \bigwedge_{x,y \in E} (x,y) \in X \times X \Longrightarrow \|f(x) - f(y)\| \leq \leq \alpha(x) \|x - y\| ,$$

(ii) $f: E \longrightarrow E$ is said to be a "uniformly generalized contraction with respect to X": $\Longleftrightarrow$

$$(**) \quad \bigvee_{\alpha: E \to [0,1)} \ \bigwedge_{x,y \in E} (x,y) \in E \times X \Longrightarrow \|f(x) - f(y)\| \leq \leq \alpha(x) \|x - y\| .$$

AMS, Primary: 47H10 Ref. Ž. 7.978.53

Remark 1:

1) Contractions in the sense of Banach are generalized contractions.

2) [4]: Let $(E, \| \|)$ be a normed linear space and suppose $\emptyset \neq X \subset E$ is open, bounded and convex; let $f: X \longrightarrow E$ be continuously (Fréchet) differentiable. Then f is a generalized contraction iff $\| f'_x \| < 1$ for all $x \in X$. A similar example may be given satisfying condition $(**)$, see [3].

Theorem 1: Let $(E, \| \|)$ be a normed linear space and suppose $\mathcal{T}$ is a Hausdorff topology for E, such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge\limits_{S \subset E} \quad S$ convex $\wedge$ S $\mathcal{T}$-compact $\Longrightarrow S$ is norm-bounded,

(iii) $\bigwedge\limits_{x \in E} \bigwedge\limits_{r \geq 0} B(x, r) := \{ y \mid y \in E \wedge \|x - y\| \leq r \} \Rightarrow B(x, r)$ is $\mathcal{T}$-closed.

Let $\emptyset \neq X \subset E$ be a convex $\mathcal{T}$-compact subset of E and suppose $f: X \rightarrow X$ is a generalized contraction.

Then: (a) There is a unique $x_0 \in X$ such that $f(x_0) = x_0$;

(b) For $x \in X$ we have $\lim\limits_{n \to \infty} \{ f^n(x) \} = x_0$ (strongly).

Proof: (a): Let $\mathcal{V} := \{ S \mid \emptyset \neq S \subset X,\ S$ convex, $\mathcal{T}$-closed and $f(S) \subset S \}$.

We have $\mathcal{T} \neq \emptyset$ $(X \in \mathcal{T})$. Ordered by $S_1 \preceq S_2 :\Longleftrightarrow$ $\Longleftrightarrow S_1 \supset S_2$, it can easily be seen, $(\mathcal{T}, \preceq)$ being inductively ordered. Let $S_0 \in \mathcal{T}$ be maximal (Zorn). Defining $\delta := diam\,(S_0)$ we have $0 \leq \delta < \infty$ (ii). Assume $\delta > 0$ and let $x \in S_0$; we define $\delta_1 :=$ $:= \alpha(x)\delta$ and $S_1 := S_0 \cap B(f(x), \delta_1)$. We have $\emptyset \neq S_1 \subset X$ $(S_0 \subset X \wedge f(x) \in S_1)$ and S_1 is $\mathcal{Z}$-closed by (iii). Finally, we have for $z \in S_1$ $f(z) \in S_0$ and $\| f(x) - f(z) \| \leq \alpha(x) \| x - z \| \leq \alpha(x)\delta \leq \delta_1$, i.e. $f(S_1) \subset S_1 : S_1 = S_0$ (maximality of S_0). This implies $S_0 \subset B(f(x), \delta_1)$. Now define $S_2 :=$ $:= \bigcap_{y \in S_0} S_0 \cap B(y, \delta_1)$. Then $\emptyset \neq S_2 \subset X$ $(S_0 \subset X \wedge f(x) \in S_2)$, S_2 is convex and $\mathcal{Z}$-closed by (iii). It is easily verified that $(*)\,\overline{co\,[f(S_0)]}^{\mathcal{Z}} =$ $= S_0$ ($\mathcal{Z}$-closed convex hull) [Take $S_3 := \overline{co\,[f(S_0)]}^{\mathcal{Z}}$ and prove $S_3 \in \mathcal{T}$ and $S_3 \subset S_0$]. Now let $u \in S_2$ and $y \in S_0$.

Then $\| f(u) - f(y) \| \leq \| u - y \| \leq \delta_1$, i.e. $f(S_0) \subset$ $\subset B(f(u), \delta_1)$. It follows $S_0 = \overline{co\,[f(S_0)]}^{\mathcal{Z}} \subset$ $\subset \overline{B(f(u), \delta_1)}^{\mathcal{Z}} \subset B(f(u), \delta_1)$ by (iii), i.e. $f(u) \in \bigcap_{y \in S_0} B(y, \delta_1) \cap S_0$, i.e. $f(u) \in S_2$. The maximality of S_0 gives $S_2 = S_0$. Finally let $u, v \in$ $\in S_2$; we have $u \in B(v, \delta_1)$ $(v \in S_0)$ implying $\| u - v \| \leq \delta_1$ and $diam\,(S_2) \leq \delta_1 < \delta = diam\,(S_2)$, a contradiction: We have $\delta = 0$, i.e. there exists $x_0 \in$ $\in X$ such that $S_0 = \{x_0\}$. Because of $f(S_0) \subset S_0$ we have $f(x_0) = x_0$; (b) Let $z \in X$ and $n \in \mathbb{N}$. Then $\| f^n(z) - x_0 \| \leq \| f^n(z) - f(x_0) \| \leq \alpha(x_0) \| f^{n-1}(z) - x_0 \|$

implying (by induction) $\|f^n(x)-x_0\| \leq [\alpha(x_0)]^n \|x-x_0\|$ such that $\lim_{n\to\infty}\{f^n(x)\}=x_0 \quad (0\leq\alpha(x_0)<1)$; (b) is proved. The uniqueness of x_0 is an immediate consequence of (b) or, directly of f's contraction property ($\|f(x)-f(y)\| \leq \|x-y\|$ for $x\neq y$.

Corollary 1: Let $(E,\|\ \|)$ be a normed linear space, let $\mathcal{T}$ be a Hausdorff topology for E with (i) - (iii) of Theorem 1. Let $R\geq 0$ and suppose $B(0,R)$ is $\mathcal{T}$-compact and $f: B(0,R)\longrightarrow E$ is a generalized contraction such that $\|f(x)\|\leq R$ if $\|x\|=$ $=R$ (i.e. $f(\mathrm{Bd}(B(0,R)))\subset B(0,R))$.

Then: (a) There exists a unique $x_0\in B(0,R)$ such that $f(x_0)=x_0$;

(b) For $x\in B(0,R)$ we have $\lim_{n\to\infty}\{[\frac{1}{2}(\mathrm{Id}+f)]^n(x)\}=x_0$ (strongly).

Proof (see [4]): Define $g: B(0,R)\longrightarrow E$ by $g:=\frac{1}{2}(\mathrm{Id}+f)$. Then we have $g(B(0,R))\subset B(0,R)$, g is a generalized contraction, the fixed point sets of f and g are the same. Theorem 1 completes the proof.

Remark 2:

Examples for $\mathcal{T}$:

1) Let $(E,\|\ \|)$ be a conjugate space and let $\mathcal{T}$ be the weak* topology for E . Then (i) - (iii) of Theorem 1 comes true.

2) Let $(E,\|\ \|)$ be a reflexive Banach space and let $\mathcal{T}$ be the weak topology for E . Then (i) - (iii) of Theorem 1 comes true.

3) W.A. Kirk [4] proves Theorem 1 and Corollary 1 in the

case of a conjugate space $(E, \| \|)$ and the weak* topology for E.

Theorem 2: Let $(E, \| \|)$ be a normed linear space, suppose $\mathcal{T}$ is a Hausdorff topology for E, such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge_{S \subset E}$ S convex $\wedge$ S $\mathcal{T}$-compact $\Rightarrow$ S is norm-bounded,

(iii) $\bigwedge_{x \in E} \bigwedge_{r \geq 0} B(x, r) := \{y \mid y \in E \wedge \|x - y\| \leq r\} \Rightarrow B(x, r)$ is $\mathcal{T}$-closed,

(iv) The norm topology for E is finer than $\mathcal{T}$.

Let $\emptyset \neq X \subset E$ be a convex $\mathcal{T}$-compact and $\mathcal{T}$-(sequentially compact) subset of E, let $f : X \longrightarrow E$ be a generalized contraction and $g : [X, \mathcal{T}] \longrightarrow [E, \| \|]$ sequentially continuous such that

$$(K_1) \quad \bigwedge_{x, y \in E} (x, y) \in X \times X \Rightarrow f(x) + g(y) \in X .$$

Then $f + g$ has a fixed point.

Proof: Let $y \in X$. We define $h_y : X \longrightarrow X$ (K_1) by $h_y(x) := f(x) + g(y)$; h_y is a generalized contraction. By Theorem 1 there is a unique $x_y \in X$ such that $h_y(x_y) = x_y$. Defining $T : X \longrightarrow X$ by $T(y) := x_y$ we have for $y, z \in X$

$$\|T(y) - T(z)\| \leq \|x_y - x_z\| \leq \|h_y(x_y) - h_z(x_z)\| \leq$$

$$\leq \|f(x_y) - f(x_z) + g(y) - g(z)\| \leq \|f(x_y) - f(x_z)\| +$$

$$+ \|g(y) - g(z)\| \leq \alpha(x_y) \|x_y - x_z\| + \|g(y) - g(z)\| \leq$$

$$\leq \alpha(x_y) \|T(y) - T(z)\| + \|g(y) - g(z)\| ,$$

such that

$$(*) \quad \|T(y)-T(x)\| \leq \frac{1}{1-\alpha(x_y)}\|g(y)-g(x)\| .$$

T is continuous in the norm topology: let $\{x_n\} \in X^{\mathbb{N}}$ and $x_0 \in X$ such that $x_n \to x_0$ (strongly). Then by (iv) $\mathcal{T}-\lim_{n\to\infty}\{x_n\} = x_0$. Now $g(x_n) \to g(x_0)$ and $\{T(x_n)\} \to T(x_0)$ (strongly) by $(*)$. Let $\{T(x_n)\} \in X^{\mathbb{N}}$, $\{x_n\} \in X^{\mathbb{N}}$. There is a subsequence $\{x_n'\} \in X^{\mathbb{N}}$ of $\{x_n\} \in X^{\mathbb{N}}$ and $x_1 \in X$ such that $\mathcal{T}-\lim_{n\to\infty}\{x_n'\} = x_1$ (X is $\mathcal{T}$ - (sequentially compact)). Then $g(x_n') \to g(x_1)$ (strongly), **consequently** by $(*)$ $\|T(x_n') - T(x_1)\| \leq \frac{1}{1-\alpha(x_{x_1})}\|g(x_n')-g(x_1)\| \to 0$, i.e. $\{T(x_n)\}$ has a (strongly) convergent subsequence. Finally X is norm-bounded (ii) and norm-closed, because X is $\mathcal{T}$-closed and $\mathcal{T}$ is coarser than the norm topology. Schauder's fixed point theorem completes the proof (for let $y \in X$ such that $y = T(y)$ then $y = T(y) = x_y$ and $x_y = h(x_y) = f(x_y) + g(y)$, i.e. $y = f(y) + g(y)$).

Remark 3:

1) W.V. Petryshyn [7] proves Theorem 2 in the case of a reflexive Banach space $(E, \|\ \|)$ and the weak topology for E (satisfying all conditions of Theorem 2) for a subset $X \subset E$ additionally satisfying $\mathrm{int}(X) \neq \emptyset$ (degree method).

2) In the case of a conjugate space $(E, \|\ \|)$ and the weak* topology for E, a $\mathcal{T}$-compact convex subset of

E need not be $\mathcal{T}$ - (sequentially compact). This, however, is true, if $(E, \| \|)$ is strongly separable ([10], p.209).

3) The Krasnoselski condition (K_1) is very restrictive, as the following simple example shows: Let $E := \mathbb{R}$ (absolute value norm), $X := [0,1]$; $f, g : X \longrightarrow E$ defined by $f(x) := \frac{1}{2}x$, $g(x) := 1 - \frac{1}{2}x$. Then $(1, 0) \in X \times X$ but $f(1) + g(0) = \frac{3}{2} \notin X$. In the case of a Banach contraction f and a compact g and a closed, bounded (strongly), convex subset $X \subset E$ (K_1) can be weakened to "$(f+g)(X) \subset X$" ([1],[8]). In our situation this could be done also (see the proof of Theorem 4), if (i) $Id - f$ is demiclosed [8], or (ii) $(Id - f)(X)$ is closed, or (iii) $(Id - f - g)(X)$ is closed, or (iv) If $0 \in \overline{(Id - f - g)(X)}^{strong}$ then $0 \in (Id - f - g)(X)$.

With the same method employed in Theorem 2 - now using Corollary 1 - we can prove

<u>Theorem 3</u>: Let $(E, \| \|)$ be a normed linear space and suppose $\mathcal{T}$ is a Hausdorff topology for E, such that

(i) $(E, \mathcal{T})$ is a topological linear space,

(ii) $\bigwedge_{S \subset E}$ S convex $\wedge$ S $\mathcal{T}$-compact $\Longrightarrow$ S is norm-bounded,

(iii) $\bigwedge_{x \in E} \bigwedge_{r \geq 0} B(x, r) := \{y \mid y \in E \wedge \|x - y\| \leq r\} \Longrightarrow B(x, r)$ is $\mathcal{T}$-closed,

(iv) The norm topology for E is finer than $\mathcal{T}$.

Let $R \geq 0$ and suppose $B(0, R)$ is $\mathcal{T}$-compact and $\mathcal{T}$ - (sequentially compact) and $f := B(0, R) \longrightarrow E$ is a generalized contraction, let

$g : [X, \mathcal{T}] \longrightarrow [E, \| \|]$ be sequentially continuous, such that

$$(K_2) \quad \bigwedge_{x, y \in E} \|x\| = R \wedge \|y\| \leq R \Rightarrow f(x) + g(y) \in B(0, R) .$$

<u>Then</u> $f + g$ has a fixed point.

<u>Remark 4:</u>

W.V. Petryshyn provës Theorem 3 in [7] in the case of a reflexive Banach space and the weak topology (see Remark 2).

The method developed in [3] yields

<u>Lemma 1</u>: Let $(E, \| \|)$ be a reflexive Banach space and suppose X is a nonvoid, closed, bounded, convex subset of E ; let $f : E \longrightarrow E$ be a uniformly generalized contraction with respect to X and $\{x_n\} \in X^{\mathbb{N}}$ such that $\lim_{n \to \infty} \{x_n - f(x_n)\} = 0$ (strongly).

<u>Then</u> (a) f has a unique fixed point $x_0 \in X$,

(b) $\lim_{n \to \infty} \{x_n\} = x_0$ (strongly).

<u>Proof</u>: See [3], proof of Theorem 2.

As a corollary of Lemma 1 we obtain

<u>Lemma 2</u>: Let $(E, \| \|)$ be a reflexive Banach space and suppose X is a nonvoid, closed, bounded, convex subset of E ; let $f : E \longrightarrow E$ be a uniformly generalized contraction with respect to X and let $\{x_n\} \in X^{\mathbb{N}}$ and $y \in E$ such that $\lim_{n \to \infty} \{x_n - f(x_n)\} = y$ (strongly).

<u>Then</u> (a) There is a unique $x_1 \in X$ such that $x_1 - f(x_1) = y$,

(b) $\lim_{n \to \infty} \{x_n\} = x_1$.

Proof: Define $g : E \longrightarrow E$ by $g(x) := -f(x) + y$. Then g is a uniformly generalized contraction with respect to X and $\lim_{n\to\infty} \{x_n - g(x_n)\} = 0$ (strongly). Thus, by Lemma 1, there is a unique $x_1 \in X$ such that $g(x_1) = x_1$, i.e. $x_1 - f(x_1) = y$ and $\lim_{n\to\infty} \{x_n\} = x_1$ (strongly).

Theorem 4: Let $(E, \|\ \|)$ be a reflexive Banach space and suppose X is a nonvoid, closed, bounded, convex subset of E; let $f : E \longrightarrow E$ be a uniformly generalized contraction with respect to X and let $g : X \longrightarrow E$ be compact such that $(f+g)(X) \subset X$.
Then $f + g$ has a fixed point.

Proof: Without loss of generality we may assume $0 \in X$. Let $\{\lambda_n\} \in (0,1)^{\mathbb{N}}$ with $\lim_{n\to\infty} \{\lambda_n\} = 1$. We define $f_n := \lambda_n f$, $g_n := \lambda_n g$ for $n \in \mathbb{N}$ and we have $(f_n + g_n)(X) \subset X$. Because of $\|f_n(x) - f_n(y)\| \le \lambda_n \alpha(x)\|x-y\| \le \lambda_n \|x-y\|$ and g_n being compact, there is a sequence $\{x_n\} \in X^{\mathbb{N}}$ such that $f_n(x_n) + g_n(x_n) = x_n$ (see [1],[8]). Because of g's compactness there exists a subsequence $\{x_n'\} \in X^{\mathbb{N}}$ of $\{x_n\}$ and $y \in E$ such that $\lim_{n\to\infty} \{g(x_n')\} = y$ (strongly). Now we have for $n \in \mathbb{N}$: $x_n' - f(x_n') - g(x_n') = (\lambda_n' - 1)(f(x_n') + g(x_n'))$. The boundedness of X implies $\lim_{n\to\infty} \{x_n' - f(x_n')\} = y$ (strongly). By Lemma 2 we have a $x_1 \in X$ with $x_1 - f(x_1) = y$ and $\lim_{n\to\infty} \{x_n'\} = x_1$ (strongly). Finally the continuity of g induces $\lim_{n\to\infty} \{g(x_n')\} = g(x_1)$ such that $y = g(x_1)$: We have $x_1 - f(x_1) = g(x_1)$, i.e. $f(x_1) + g(x_1) = x_1$, q.e.d.

The same method used in the proof of Theorem 4 yields

Theorem 5: Let $(E, \|\ \|)$ be a reflexive Banach space and suppose $\mathcal{X}$ is a closed, bounded, convex subset of E and $x_0 \in int(\mathcal{X})$; let $f : E \to E$ be a uniformly generalized contraction with respect to $\mathcal{X}$ and $g : \mathcal{X} \to$ $\to E$ be such that

$$(K_3) \bigwedge_{x,y \in E} \bigwedge_{\lambda \in R} x \in bd\,(\mathcal{X}) \wedge (f+g)(x) = \lambda x + (1-\lambda)x_0 \Longrightarrow \lambda \leq 1 .$$

Then $f + g$ has a fixed point.

Remark 5:

Theorem 5 is proved by W.A. Kirk in [3] for $x_0 = 0$ (using a method of F.E. Browder [2]) and by W.V. Petryshyn in [7] (degree method).

References

[1] S. FUČÍK: Fixed point theorems for sum of nonlinear mappings, Comment.Math.Univ.Carolinae 9(1968), 133-143.

[2] F.E. BROWDER: Semicontractive and semiaccretive nonlinear mappings in Banach spaces, Bull.Amer.Math. Soc. 74(1968),660-665.

[3] W.A. KIRK: On nonlinear mappings of strongly semicontractive type, J.Math.Anal.Appl.27(1969),409-412.

[4] W.A. KIRK: Mappings of generalized type, J.Math.Anal. Appl.32(1970),567-572.

[5] M.A. KRASNOSELSKI: Two remarks on the method of successive approximations, Uspehi Mat.Nauk 10,No 1(63)(1955),123-127.

[6] W.V. PETRYSHYN: Structure of the fixed point sets of k-set-contractions, Arch.Rat.Mech.Anal.40(1971), 312-328.

[7] W.V. PETRYSHYN: A new fixed point theorem and its application, Bull.Amer.Math.Soc.78(1972), 225-229.

[8] J. REINERMANN: Fixpunktsätze vom Krasnoselski-Typ, Math.Z.119(1971),339-344.

[9] J. REINERMANN: Über Fixpunkte kontrahierender Abbildungen in uniformen Räumen und deren Darstellung durch konvergente Iterationsverfahren, Ber.d.Ges.f.Math.u.Datenverarb.Bonn,Nr 4, 1968.

[10] A.E. TAYLOR: Introduction to Functional Analysis, John Wiley & Sons,Inc.,London-Sydney,sixth printing 1967.

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(Oblatum 27.7.1972)