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EXTENDING TENSOR PRODUCTS TO STRUCTURES OF CLOSED CATEGORIES

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Let $\mathfrak{K}$ be a category, I an object, $\otimes : \mathfrak{K} \times \mathfrak{K} \rightarrow \mathfrak{K}$, $H : \mathfrak{K}^{op} \times \mathfrak{K} \rightarrow \mathfrak{K}$ functors such that $\mathfrak{K}(A \otimes B, C) \cong \mathfrak{K}(A, H(B, C))$ and $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ naturally in A, B, C, $A \otimes B \cong B \otimes A$ naturally in A, B and $A \otimes I \cong A$ naturally in A. The natural equivalence being unspecified, the problem arises whether they may be chosen coherent in the sense of MacLane, in other words, whether the collection of data $(\otimes, H, I)$ can be extended to a structure of a closed category on $\mathfrak{K}$ (in the sense of [2] - symmetric monoidal closed in the sense of [1]).

In the present paper, this question is positively answered (Theorem 4.4) for the case where I is a generator of $\mathfrak{K}$. Moreover, in this case it is shown that the associativity and commutativity equivalences are uniquely determined by the data $(\otimes, H, I)$ and the variety of the remaining information described (Theorems 5.3 and 5.7)

The condition on I to be a generator is certainly restrictive and the author has to admit he does not know whether it is essential at all. No counterexample is known, i.e.,

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all the known systems $(\otimes, H, I)$ with non-generating I are parts of structures of a closed category, but the attempts to prove a general extension theorem were so far unsuccessful . On the other hand, in the case of concrete categories $(\mathfrak{K}, \mathfrak{U})$ with $\mathfrak{U} : \mathfrak{K} \longrightarrow \mathcal{S}et$ faithful (which leaves out some important cases, e.g. the category of small categories with the discretization for $\mathfrak{U}$) and those $(\otimes, H, I)$, where H behaves like a hom-functor, i.e. $\mathfrak{U}\ H \cong \mathfrak{K}(-,-)$, I is necessarily a generator (see 1.4 2)), so that here the result holds without restriction.

Acknowledgment. I am indebted to Professor Saunders MacLane for stimulating this paper and valuable advice.

§ 1. Preliminaries

1.1. Definition. A preclosed category is a category $\mathfrak{K}$ together with a fixed object I , functors $\otimes : \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathfrak{K}$ and $H : \mathfrak{K}^{op} \times \mathfrak{K} \longrightarrow \mathfrak{K}$, and natural equivalences

$$a^{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C) ,$$

$$r^{A} : A \otimes I \longrightarrow A ,$$

$$c^{AB} : A \otimes B \longrightarrow B \otimes A ,$$

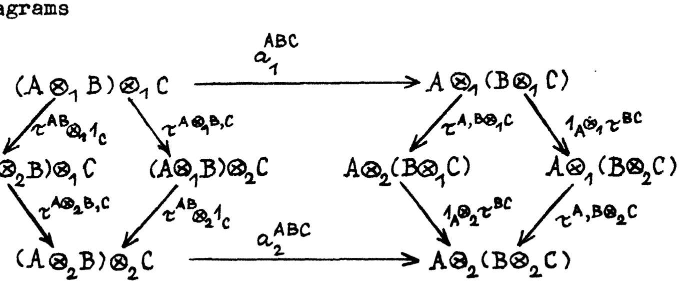
$$k^{ABC} : \mathfrak{K}(A \otimes B, C) \longrightarrow \mathfrak{K}(A, H(B, C)) .$$

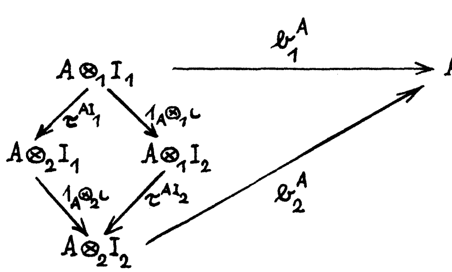
It is said to be a closed category if, moreover, $(\otimes, I, a, r, c)$ is a coherent multiplication in the sense of MacLane ([4],[5]). The collection of data $(\otimes, H, I, a, r, c, k)$ is called a structure of a closed (preclosed, resp.) category on $\mathfrak{K}$. Abbreviated, SC (SPC, resp.).

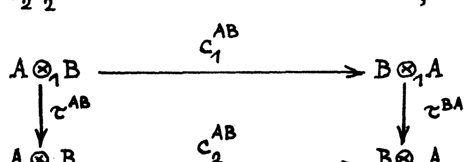
If a functor $\otimes : \mathfrak{K} \times \mathfrak{K} \longrightarrow \mathfrak{K}$ (couple $(\otimes, H,)$, triple $(\otimes, H, I)$ resp.) can be extended to an SPC, it is called a <u>tensor product</u> (<u>tensor couple</u>, <u>tensor structure</u>, resp.) on $\mathfrak{K}$.

The object I is called a unit of $\otimes$.

Two PSC $(\otimes_i, H_i, a_i, b_i, c_i, k_i)$ $(i = 1, 2)$ are said to be equivalent if there exists a natural equivalence $\tau : \otimes_1 \longrightarrow \otimes_2$ and an isomorphism $\iota : I_1 \longrightarrow I_2$ such that the diagrams

(E1)  ,

(E2)  ,

(E3)  ,

commute.

1.2. <u>Remark</u>. This definition of a closed category differs only formally from that of [2]. By the construction described in [2] immediately after the definition we see readily that the SCs in the two senses are in a one-to-one correspondence.

1.3. <u>Remark</u>. If $(\otimes, H, I, a, b, c, \mathfrak{K})$ is an SPC then $H(I, X) \cong X$ naturally in X. Really, we have $\mathfrak{K}(Y, H(I, X)) \cong \mathfrak{K}(Y \otimes I, X) \cong \mathfrak{K}(Y, X)$ naturally in Y, X which yields the statement.

1.4. <u>Definitions and remarks</u>. 1) Given a concrete category $(\mathfrak{K}, \mathfrak{U})$ (a category $\mathfrak{K}$ with a fixed functor $\mathfrak{U} : \mathfrak{K} \longrightarrow \mathit{Set}$; the functor $\mathfrak{U}$ is mostly - but not always - assumed faithful), a tensor product on $(\mathfrak{K}, \mathfrak{U})$ is a tensor product on $\mathfrak{K}$ such that, for the associated H, $\mathfrak{U}(H(A, B)) \cong \mathfrak{K}(A, B)$ naturally in A, B. In this sense we also speak about a tensor couple, tensor structure, SC on $(\mathfrak{K}, \mathfrak{U})$.

2) If there is a tensor structure $(\otimes, H, I)$ on $(\mathfrak{K}, \mathfrak{U})$ then $\mathfrak{U}$ is naturally equivalent to $\mathfrak{K}(I, -)$ (we have $\mathfrak{U}(X) \cong \mathfrak{U}H(I, X) \cong \mathfrak{K}(I, X)$ by 1.3). Thus, if the $\mathfrak{U}$ is assumed faithful, I is necessarily a generator.

3), Obviously, $\otimes$ determines the I up to isomorphism.

4) We shall see later (in 2.2) that the unit has to have commutative endomorphism semigroup. Comparing this with 2) we see that the choice of a forgetful functor $\mathfrak{U}$ on $\mathfrak{K}$ such that $(\mathfrak{K}, \mathfrak{U})$ has a tensor product is usually rather

restricted. See, however, 2.9.

1.5. Proposition. An SPC equivalent to an SC is an SC.

Proof. Just a tedious checking of the coherence properties.

1.6. Proposition. Let $(\otimes, H, I, a, b, c, k)$ be an SPC such that $c^{II} = 1_{I\otimes I}$ and $c^{IA} . c^{AI} = 1_{I\otimes A}$. Then there is an equivalent one $(\otimes', H, I, a', b', c', k')$ such that $b' = 1$ and $c'^{AI} = c'^{IA} = 1_A$ (and, hence, always $1_I \otimes' \varphi = \varphi \otimes' 1_I = \varphi$).

Proof. Put $A \otimes' B = A \otimes B$ for $A, B \neq I$, $A \otimes' I = I \otimes' \otimes' A = A$, define $\tau^{AB} : A \otimes B \longrightarrow A \otimes' B$ by $\tau^{AB} = 1$ for $A, B \neq I$, $\tau^{AI} = b^A$, $\tau^{IA} = b^A c^{IA}$, and, for $\alpha : A \to C$, $\beta : B \longrightarrow D$, put $\alpha \otimes' \beta = \tau^{CD} . (\alpha \otimes \beta) . \bar{\tau}^{AB}$. Obviously $\otimes'$ is a functor and $\tau : \otimes \longrightarrow \otimes'$ a natural equivalence. Now, it suffices to put

$$a'^{ABC} = \tau^{A,B\otimes' C} . (1_A \otimes \tau^{BC}) . a^{ABC} . (\bar{\tau}^{AB} \otimes 1_C) . \bar{\tau}^{A\otimes' B, C} ,$$

$$b'^A = b^A . \bar{\tau}^{AI} , \quad c'^{AB} = \tau^{BA} . c^{AB} . \bar{\tau}^{AB}$$

(the bars designate inverses).

§ 2. More about the unit I

Throughout this paragraph a preclosed category $(\mathcal{K}, \otimes, H, I, a, b, c, k)$ is assumed to be given. If there is no danger of confusion, a^{III}, b^I, c^{II} are written simply a, b, c.

2.1. Lemma. For every $\alpha : I \longrightarrow I$ there is exactly one $\alpha' : I \longrightarrow I$ with $\alpha \otimes 1_I = 1_I \otimes \alpha'$ (and vice versa).

Proof. Namely, $\alpha' = \mathfrak{b} . (1 \otimes \alpha) . \bar{\mathfrak{b}}$. Then we have $1 \otimes \alpha' = \bar{c}\bar{\mathfrak{b}}\mathfrak{b}c(1 \otimes \alpha') = \bar{c}\bar{\mathfrak{b}}\mathfrak{b}(\alpha' \otimes 1)c = \bar{c}\bar{\mathfrak{b}}\alpha'\mathfrak{b}c = \bar{c}(1 \otimes \alpha)c = \alpha \otimes 1$. The unicity is obvious.

2.2. *Lemma*. For any two $\alpha, \beta : I \longrightarrow I$, $\alpha\beta = \beta\alpha$.

Proof. We have $\alpha\beta = \mathfrak{b}(\alpha \otimes 1)\bar{\mathfrak{b}}\mathfrak{b}(\beta \otimes 1)\bar{\mathfrak{b}} = \mathfrak{b}(\alpha \otimes 1)(1 \otimes \beta')\bar{\mathfrak{b}} = \mathfrak{b}(1 \otimes \beta')(\alpha \otimes 1)\bar{\mathfrak{b}} = \beta\alpha$.

2.3. *Lemma*. For any two $\alpha, \beta : I \longrightarrow I$,

$$\alpha \otimes \beta = \beta \otimes \alpha = 1_I \otimes (\alpha\beta) = (\alpha\beta) \otimes 1_I .$$

In particular, $\alpha \otimes 1_I = 1_I \otimes \alpha$.

Proof. Since $I \otimes I \cong I$, the morphisms $I \otimes I \longrightarrow \to I \otimes I$ also commute. Thus, $\alpha \otimes \beta = \bar{c}(\beta \otimes \alpha)\bar{c} = \bar{c}c(\beta \otimes \alpha) = \beta \otimes \alpha$. Consequently, $\alpha \otimes \beta = (\alpha \otimes 1)(\beta \otimes 1) = (\alpha \ \beta) \otimes 1$.

2.4. *Lemma*. For every morphism (isomorphism, resp.) $\gamma : I \longrightarrow I$ there is a natural transformation (natural equivalence, resp.) $\tau : 1_{\mathfrak{K}} \longrightarrow 1_{\mathfrak{K}}$ with $\tau^I = \gamma$. If I is a generator, there is exactly one such τ .

Proof. Put $\tau^A = \mathfrak{b}^A . (1_A \otimes \gamma) . \bar{\mathfrak{b}}^A$. The unicity for the case of I a generator is evident.

2.5. *Lemma*. A $\otimes$ -iterate is a functor obtained recursively by the following rules:

(i) $\otimes$ is a $\otimes$ -iterate, $1_{\mathfrak{K}}$ is a $\otimes$ -iterate,

(ii) if $F_1, \ldots, F_n$, F are $\otimes$ -iterates, F in n variables, $F \circ (F_1 \times \ldots \times F_n)$ is an $\otimes$ -iterate. Generalized $\otimes$ -iterates are obtained from $\otimes$ -iterates by permuting the variables and replacing some of them by constants.

Let F, G be generalized $\otimes$ -iterates, $\tau, \vartheta : F \longrightarrow$ $\longrightarrow G$ natural transformations. Let I be a generator. Then $\tau = \vartheta$ iff $\tau^{I\ldots I} = \vartheta^{I\ldots I}$.

Proof. Let $\alpha : I \longrightarrow A$, $\varphi : A \otimes B \longrightarrow C$ be morphisms. We have

$$\mathscr{k}(\varphi \cdot (\alpha \otimes 1_B)) = (\mathscr{k} \cdot \mathscr{H}(\alpha \otimes 1, 1))(\varphi) =$$
$$= \mathscr{H}(\alpha, H(1,1))(\mathscr{k}(\varphi)) = \mathscr{k}(\varphi) \cdot \alpha \ .$$

Thus, we have

(1) $(\forall \alpha : I \to A \quad \varphi \cdot (\alpha \otimes 1) = \psi \cdot (\alpha \otimes 1)) \Longrightarrow \varphi = \psi$.

Using the natural equivalence c we obtain

(2) $(\forall \alpha : I \to A \quad \varphi \cdot (1 \otimes \alpha) = \psi \cdot (1 \otimes \alpha)) \Longrightarrow \varphi = \psi$.

Hence, since $\varphi \cdot (\alpha \otimes \beta) = \varphi \cdot (\alpha \otimes 1) \cdot (1 \otimes \beta)$,

(3) $(\forall \alpha : I \to A, \beta : I \to B \quad \varphi \cdot (\alpha \otimes \beta) = \psi \cdot (\alpha \otimes \beta)) \Longrightarrow \varphi = \psi$.

Now, we easily obtain by induction that for a generalized $\otimes$ -iterate F

$$(\forall \alpha_i : I \to A_i \quad \varphi \cdot F(\alpha_1, \ldots, \alpha_n) = \psi \cdot F(\alpha_1, \ldots, \alpha_n)) \Longrightarrow \varphi = \psi ,$$

from which the statement immediately follows.

2.6. Since a, b, c are natural equivalences, we obtain immediately

Lemma. 1) $b^{I \otimes I} = b^I \otimes 1$.

2) $a^{I \otimes I, I, I} = (\bar{b}^I \otimes 1_{I \otimes I}) \cdot a^{III} \cdot ((b^I \otimes 1_I) \otimes 1_I)$.

3) $a^{I, I \otimes I, I} = (1_I \otimes (\bar{b}^I \otimes 1_I)) \cdot a^{III} \cdot ((1_I \otimes b^I) \otimes 1_I)$.

4) $a^{I, I, I \otimes I} = (1_I \otimes (1_I \otimes \bar{b}^I)) \cdot a^{III} \cdot (1_{I \otimes I} \otimes b^I)$.

5) If $c^{II} = 1_{I\otimes I}$ then $c^{I,I\otimes I} = \bar{b}^{I} \otimes b^{I}$.

2.7. <u>Theorem</u>. Let I be a generator of $\mathfrak{K}$, let $(\otimes, H, I, a, b, c, k)$ be an SPC on $\mathfrak{K}$. Then it is an SC iff $a^{III} = b^{I} \otimes \bar{b}^{I}$ and $c^{II} = 1_{I\otimes I}$.

<u>Proof</u>. We shall use the notation of coherence requirements from [2] (C1 – C4). a^{III} , b^{I} , c^{II} shall be abbreviated to a, b, c resp. Let $(\otimes, H, I, a, b, c, k)$ be an SC. Then, $a = b \otimes \bar{b}$ is obtained immediately from C2 . Further, by C4 we obtain

$$a . c^{I,I\otimes I} . a = (1 \otimes c) . a . (c \otimes 1) .$$

By C2 , $a . c^{I,I\otimes I} . a = (b \otimes \bar{b}) . c^{I,I\otimes I} . (b \otimes \bar{b}) = c^{I\otimes I,I}$, by C2 and 2.6.1),

$$(1 \otimes c) . a . (c \otimes 1) = (1 \otimes c) . (1 \otimes \bar{b}) . b^{I\otimes I} . (c \otimes 1) =$$
$$= (1 \otimes c) . (1 \otimes \bar{b}) . c . b^{I\otimes I} = (1 \otimes c) . c^{I\otimes I,I} .$$

Thus $1 \otimes c = 1$, so that $c = 1$.

On the other hand, let $a = b \otimes \bar{b}$ and $c = 1$. By 2.5 it suffices to check C1 – C4 at the values $I, \dots, I$. By 2.6.2) – 4) we have

$$(1 \otimes a) . a^{I,I\otimes I,I} . (a \otimes 1) =$$
$$= (1 \otimes (1 \otimes \bar{b})) . (b \otimes \bar{b}) . ((b \otimes 1) \otimes 1) =$$
$$= (1 \otimes (1 \otimes \bar{b})) . (b \otimes \bar{b}) . (1 \otimes b) . (\bar{b} \otimes 1) . (b \otimes \bar{b}) . ((b \otimes 1) \otimes 1) =$$
$$= a^{I,I,I\otimes I} . a^{I\otimes I,I,I}$$

which gives C1 ; C2 is required in $a = b \otimes \bar{b}$ by 2.6.1), C3 is trivial. Finally, we have

$(1\otimes c).a.(c\otimes 1) = b\otimes\bar{b} = (b\otimes\bar{b}).(\bar{b}\otimes b).(b\otimes\bar{b}) = a.c^{I,I\otimes I}.a$

by 2.6.5), so that also C4 holds.

2.8. Remarks. 1) By the proof of 2.7 we see that in the case of a generator I, C2 and C4 imply C1 and C3 .

2) If I has no non-identical automorphism, then every SPC $(\otimes, H, I, a, b, c, k)$ is an SC. Moreover, the natural equivalences a, b, c are uniquely determined by $\otimes, H, I$. We shall see later (5.6) that also the natural equivalence k is uniquely determined.

2.9. Remark. Lemma 2.2 often limits radically the candidates for units of possible SCs on a given category. We will show now elementary examples of categories with many objects starting an SC as a unit. Take a partially ordered set $(X, \leq)$. Regarding it as a category in the usual way, we see easily that an SC on $(X, \leq)$ consists of two binary operations $\otimes$ and H on X such that $(X, \leq, \otimes)$ is a partially ordered commutative monoid and

(1) $\quad x\otimes y \leq z \quad$ iff $\quad x \leq H(y, z)$.

Thus, e.g., any $\otimes$ such that $(X, \leq, \otimes)$ is a partially ordered abelian group makes an SC with $H(y, z) = (-y)\otimes z$. In particular, for a discrete category, any structure of an abelian group is an SC (and vice versa: the condition (1) gives here $x\otimes y = z$ iff $x = H(y, z)$, so that, denoting by i the unit, we obtain $x\otimes H(x, i) = i$) and hence any of its objects is a unit of an SC.

This is, however, a too trivial example. To give a better one, take a linearly ordered $(X, \leq)$ with a smallest element 0 and a largest element 1, and an $e \in X$, $e \neq 0$. Put $x \otimes 0 = 0 \otimes x = 0$, for $x \leq e$ and $y \leq e$ put $x \otimes y = \min(x, y)$, otherwise $x \otimes y = \max(x, y)$. Put $H(0, x) = 1$, for $0 < y \leq x \leq e$ put $H(y, x) = e$, for $e < y$ and $x < y$ put $H(y, x) = 0$, otherwise $H(y, x) = x$. It is easy to check that this is an SC on $(X, \leq)$. Thus, taking a complete linear ordering with smallest and largest elements, we have an example of a complete cocomplete category such that every object except cosingleton (= initial object) is a unit of an SC (since $- \otimes X$ is a left adjoint, a cosingleton can be a unit only in the category with a single morphism).

§ 3. <u>Equivalence of SC with generators as units</u>

3.1. <u>Lemma</u>. Let $\mathcal{S}_i = (\otimes_i, H_i, I_i, a_i, b_i, c_i, k_i)$ $(i = 1, 2)$ be SC, let I_1 be a generator. Then $\mathcal{S}_1$ is equivalent to $\mathcal{S}_2$ iff there exists a natural equivalence $\tau : \otimes_1 \longrightarrow \otimes_2$ and an isomorphism $\gamma : I_1 \longrightarrow I_2$ such that

$$b_2^{I_2} \tau^{I_2 I_2} . (1_{I_2} \otimes_1 \gamma) = b_1^{I_2} .$$

<u>Proof</u>. Write $I = I_1$, $J = I_2$. We obtain (using 2.7)

$$a_2^{III} . (\tau^{II} \otimes_2 1_I) . \tau^{I \otimes_1 I, I} =$$

$$= (\bar{\gamma} \otimes_2 (\bar{\gamma} \otimes_2 \bar{\gamma}))(b_2^J \otimes_2 \bar{b}_2^J)((\gamma \otimes_2 \gamma) \otimes_2 \gamma)(\tau^{II} \otimes_2 1_J) .$$

$$. \tau^{I \otimes_1 I, I} = (\bar{\gamma} \otimes_2 (\bar{\gamma} \otimes_2 \bar{\gamma}))(1_J \otimes_2 \bar{b}_2^J)((b_2^J \tau^{JJ}(1_J \otimes_1 \gamma)(\gamma \otimes_1 1_I) \otimes_2$$

$\otimes_2 1_J)(1\otimes_2\gamma)\cdot\tau^{I\otimes_1 I,I} =$

$$= (\bar\gamma\otimes_2(\bar\gamma\otimes_2\bar\gamma))(1\otimes_2\bar{b}_2^J)((b_1^J(\gamma\otimes_1 1_I))\otimes_2 1_J)\tau^{I\otimes_1 I,J}(1\otimes_1\gamma) =$$

$$= \tau^{I,I\otimes_1 I}(\bar\gamma\otimes_1(\bar\gamma\otimes_2\bar\gamma))(1\otimes_1\bar{b}_2^J)((\gamma b_1^I)\otimes_1 1_J)(1\otimes_1\gamma) =$$

$$= \tau^{I,I\otimes_2 I}(b_1^I\otimes_1((\bar\gamma\otimes_2\bar\gamma)\bar{b}_2^J\gamma)) =$$

$$= \tau^{I,I\otimes_2 I}(b_1^I\otimes_1((\bar\gamma\otimes_2\bar\gamma)\bar\tau^{JJ}(1\otimes_1\gamma)\bar{b}_1^J\gamma)) =$$

$$= \tau^{I,I\otimes_2 I}(1\otimes_1\tau^{II})(b_1^I\otimes_1\bar{b}_1^I) = \tau^{I,I\otimes I}(1\otimes_1\tau^{II})a^{III} \quad ,$$

so that, by 2.5, E1 commutes. The commutativity of E2 is obtained immediately from the assumption on γ. Finally, E3 commutes since

$$\tau^{II}\cdot c_1^{II} = \tau^{II} = (\bar\gamma\otimes_2\bar\gamma)\cdot c_2^{JJ}\cdot(\gamma\otimes_2\gamma)\cdot\tau^{II} = c_2^{II}\cdot\tau^{II} \; .$$

3.2. Theorem. Let $\mathcal{S}_i = (\otimes_i, H_i, I_i, a_i, b_i, c_i, k_i)$ $(i = 1,2)$ be SC, let I_1 be a generator. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are equivalent iff $\otimes_1$ and $\otimes_2$ are naturally equivalent.

Proof. Let $\tau : \otimes_1 \longrightarrow \otimes_2$ be a natural equivalence. Put (again, $I = I_1$, $J = I_2$) $\gamma = b_1^J\cdot\tau^{JI}\cdot c_2^{IJ}\cdot\bar{b}_2^I$. Then we have

$$b_2^J\cdot\tau^{JJ}\cdot(1_J\otimes_1\gamma) = b_2^J(1_J\otimes_2\gamma)\tau^{JI} = b_2^J c_2^{JJ}(1_J\otimes_2\gamma)\tau^{JI} =$$

$$= b_2^J(\gamma\otimes_2 1_J)c_2^{JI}\tau^{II} = \gamma b_2^I c_2^{JI}\tau^{JI} = b_1^J \quad ,$$

so that the statement follows by 3.1.

§ 4. Extending a tensor product with unit a generator to a structure of closed category

4.1. Lemma. For every tensor structure $(\otimes, H, I)$ there is a natural equivalence $c^{AB}: A\otimes B \longrightarrow B\otimes A$ with $c^{II} = 1_{I\otimes I}$.

Proof. Take an SPC $(\otimes, H, I, a, b, c', k)$. Put $\varphi = b^I \bar{c}'^{II} \bar{b}^I$. Thus, $\bar{c}'^{II} = \varphi \otimes 1_I$. Further, define $\tau : 1_{\mathcal{K}} \longrightarrow 1_{\mathcal{K}}$ by $\tau^A = b^A(1_A \otimes \varphi)\bar{b}^A$ and finally $c^{AB} : A\otimes B \longrightarrow B\otimes A$ by $c^{AB} = (\tau^B \otimes 1_A)\cdot c'^{AB}$. Obviously, c is a natural equivalence. We have $\tau^I = b^I.(1_I \otimes \varphi).\bar{b}^I = b^I.(\varphi \otimes 1).\bar{b}^I = \varphi$ by 2.3, so that $c^{II} = (\varphi \otimes 1_I)c'^{II} = \bar{c}'^{II} c'^{II} = 1$.

4.2. Lemma. Let $(\otimes, H, I)$ be given, let $\otimes'$ be naturally equivalent to $\otimes$, let $\mathcal{S}' = (\otimes', H', I', a', b', c', k')$ be an SPC. Then there is an SPC $(\otimes, H, I, a, b, c, k)$ equivalent to $\mathcal{S}'$.

Proof is trivial.

4.3. Lemma. For every $\psi : H(I,X) \longrightarrow H(I,Y)$ there is a $\varphi : X \longrightarrow Y$ with $\psi = H(1_I, \varphi)$.

Proof. Put $i^X = k^{XIX}(1_X) : X \longrightarrow H(I,X)$. We see easily that thus a natural equivalence $i : 1_{\mathcal{K}} \longrightarrow H(I,-)$ is obtained. Now, it suffices to put $\varphi = \bar{i}^Y.\psi.i^X$.

4.4. Theorem. Every tensor structure $(\otimes, H, I)$ such that I is a generator can be extended to a structure of closed category.

Proof. Let $(\otimes, H, I, a, b, c, k)$ be an SPC extending $(\otimes, H, I)$. We may assume that $b = 1$ and $c^{AI} = c^{IA} = 1_A$ (Really, by 4.1, c can be chosen with $c^{II} = 1_{I\otimes I}$. Then, by 1.6, $(\otimes, H, I, a, b, c, k)$ can be replaced by an equivalent $(\otimes', H', I', a', b', c', k')$ satisfying $b' = 1$ and $c'^{AI} = c'^{IA} = 1_A$. Now, if $(\otimes', H', I')$ can be extended to an SC, $(\otimes, H, I)$ can, by 4.2 and 1.5.) Consider the diagram

$$
\begin{array}{ccc}
\mathfrak{H}(A\otimes(B\otimes C),D) & \xrightarrow{\mathfrak{H}(\alpha^{ABC},1)} & \mathfrak{H}((A\otimes B)\otimes C,D) \\
\downarrow{\scriptstyle k^{I,A\otimes(B\otimes C),D}} & & \downarrow{\scriptstyle k^{I,(A\otimes B)\otimes C,D}} \\
\mathfrak{H}(I,H(A\otimes(B\otimes C),D)) & \xrightarrow{\mathfrak{H}(1,H(\alpha^{ABC},1))} & \mathfrak{H}(I,H((A\otimes B)\otimes C,D)) \\
\downarrow{\scriptstyle \mathfrak{H}(1,\varkappa^{A,B\otimes C,D})} & & \downarrow{\scriptstyle \mathfrak{H}(1,\varkappa^{A,B,H(C,D)}\cdot\varkappa^{A\otimes B,C,D})} \\
\mathfrak{H}(I,H(A,H(B\otimes C,D))) & \xrightarrow{\mathfrak{H}(1,H(1,\varkappa^{BCD}))} & \mathfrak{H}(I,H(A,H(B,H(C,D)))) \\
\downarrow{\scriptstyle \bar{k}^{I,A,H(B\otimes C,D)}} & & \downarrow{\scriptstyle \bar{k}^{I,A,H(B,H(C,D))}} \\
\mathfrak{H}(A,H(B\otimes C,D)) & \xrightarrow{\mathfrak{H}(1,\varkappa^{BCD})} & \mathfrak{H}(A,H(B,H(C,D)))
\end{array}
$$

where $\varkappa$ is a natural equivalence $H(-\otimes -, -) \cong$ $\cong H(-, H(-, -))$ (which exists due to the associativity of $\otimes$ - this fact was first observed by Linton) and α is the transformation conjugate to $\varkappa$. Thus,

$$\alpha^{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$$

is a natural equivalence.

The big rectangle commutes by the definition of α , the outer squares commute since k is a transformation. Thus, since all the mappings involved are one-to-one onto, the inner square commutes. Since I is a generator, we obtain

(1) $\varkappa^{A,B,H(C,D)} . \varkappa^{A\otimes B,C,D} . H(\alpha^{ABC}, 1_D) =$

$$= H(1, \varkappa^{BCD}) . \varkappa^{A,B\otimes C,D} .$$

Write $\varkappa$ for $\varkappa^{III}$. Thus, $\varkappa : H(I,I) \longrightarrow H(I,H(I,I))$ and hence, by 4.3, there is a $\lambda : I \longrightarrow H(I,I)$ with $\varkappa = H(1,\lambda)$. Hence, we obtain $H(1,\varkappa) . \varkappa =$

$= H(1,H(1,\lambda)) . \varkappa = \varkappa^{I,I,H(I,I)} . H(1,\lambda) = \varkappa^{I,I,H(I,I)} . \varkappa$.

Thus, by (1), $H(\alpha^{III}, 1) = 1$, so that $\alpha^{III} = 1 =$

$= 1 \otimes 1 = \ell^{I} \otimes \bar{\ell}^{I}$. Hence, by 2.7,

$(\otimes, H, I, \alpha, \ell, c, \varkappa)$ is an SC.

4.5. Corollary. If I is a generator of $\mathfrak{K}$ then the natural equivalence classes of tensor products on $\mathfrak{K}$ with unit I are in a one-to-one correspondence with the equivalence classes of SC with unit I on $\mathfrak{K}$.

Proof. Follows immediately by 4.4 and 3.2.

4.5. Recalling 1.4 we obtain

Corollary. Let $(\mathfrak{K}, \mathfrak{U})$ be a concrete category (with $\mathfrak{U}$ faithful). Then every tensor product on $(\mathfrak{K}, \mathfrak{U})$ can be extended to an SC and thus the equivalence classes of tensor products on $(\mathfrak{K}, \mathfrak{U})$ are put in a one-to-one correspondence with the equivalence classes of SC.

4.7. Remark. A concrete category with a tensor product differs from the autonomous category of Linton ([3]) - abbreviated AC - in the following points: 1) $\mathfrak{U} \circ H$ is assumed just equivalent, not identical, with $\mathfrak{K}(-,-)$, 2) In AC the existence of unit is not assumed (if $\mathfrak{U}$ is induced by a generator, however, this I is a unit), 3) In AC a strong assumption (A5) on behavior of underlying sets and mappings is done . It has no counterpart in $(\mathfrak{K}, \mathfrak{U})$ with tensor product (except that here the commutativity of $\otimes$, which is in AC a consequence of the axioms, has to be assumed explicitly) .

In [3], 2.5, the tensor product of an AC is proved to be (associativity and commutativity) coherent, the proof depends, however, heavily on (A5).

§ 5. <u>How far a structure of a closed category is determined by a tensor product</u>

5.1. <u>Lemma.</u> Let $\otimes$ be a tensor product, $\ell^A : A \otimes I \longrightarrow A$, $\beta^A : A \otimes J \longrightarrow A$ natural equivalences. Let I be a generator. Then there is a uniquely determined isomorphism $\gamma : J \longrightarrow I$ such that $\beta^A = \ell^A . (1_A \otimes \gamma)$. On the other hand, let ℓ be given, $\gamma : J \longrightarrow I$ an isomorphism. Then $\beta^A = \ell^A . (1_A \otimes \gamma)$ is a natural equivalence.

<u>Proof</u>. If $\beta^A = \ell^A . (1_A \otimes \gamma)$ then in particular $1_I \otimes \gamma = \bar{\ell}^I . \beta^I$ and hence $\gamma = \ell^I . (\gamma \otimes 1_I) . \bar{\ell}^I = \ell^I . c^{II} . \bar{\ell}^I . \beta^I . \bar{c}^{IJ} . \bar{\ell}^J$. Evidently, for any γ, $\ell^A . (1_A \otimes \gamma)$ is a natural equivalence. Taking the γ given by the formula above, we have $\ell^I . (1_I \otimes \gamma) = \beta^I$ and hence $\ell^A . (1_A \otimes \gamma) = \beta^A$ by 2.5.

5.2. <u>Lemma</u>. Let $\ell^A : A \otimes I \longrightarrow A$, $\beta^A : A \otimes J \longrightarrow A$, $a^{ABC} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$ be natural equivalences, let $a^{III} = b^I \otimes \bar{b}^I$. Then $a^{JJJ} = \beta^J \otimes \bar{\beta}^J$.

Proof. By 5.1, $\beta^I = b^I . (1_I \otimes \gamma)$. Consequently,

$$\bar{\gamma} . b^I . (\gamma \otimes \gamma) = \bar{\gamma} . b^I . (1_I \otimes \gamma) . (\gamma \otimes 1_J) = \bar{\gamma} . \beta^I . (\gamma \otimes 1_J) = \beta^J .$$

Thus, $a^{JJJ} = (\bar{\gamma} \otimes (\bar{\gamma} \otimes \bar{\gamma})) . a^{III} . ((\gamma \otimes \gamma) \otimes \gamma) =$

$= (\bar{\gamma} . b^I . (\gamma \otimes \gamma)) \otimes \overline{(\bar{\gamma} . b^I . (\gamma \otimes \gamma))} = \beta^J \otimes \bar{\beta}^J$.

5.3. Theorem. Let $\otimes$ be a tensor product such that some (and, hence, each) of its units is a generator. Then there is exactly one natural equivalence $a^{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$ and exactly one natural equivalence $c^{AB} : A \otimes B \longrightarrow B \otimes A$ such that $(\otimes, H, I, a, b, c, k)$ is an SC for some H, I, b, k . On the other hand, I can be replaced by an arbitrary isomorphic J , and b by an arbitrary natural equivalence $\beta^A : A \otimes J \longrightarrow A$.

Proof. Let $(\otimes, H_1, I, a_1, b_1, c_1, k_1)$, $(\otimes, H_2, J, a_2, b_2, c_2, k_2)$ be two SC. Thus, $a_1^{III} = b_1^I \otimes \bar{b}_1^I$ and hence, by 5.2, $a_1^{JJJ} = b_2^J \otimes \bar{b}_2^J = a_2^{JJJ}$. Thus, $a_1 = a_2$ by 2.5. Similarly, $c_1 = c_2$, since $c_1^{JJ} = (\gamma \otimes \gamma) c_1^{II} (\bar{\gamma} \otimes \bar{\gamma}) = 1 = c_2^{JJ}$.

5.4. Corollary. A tensor structure $(\otimes, H, I)$ together with a natural equivalence $k^{ABC} : \mathfrak{K}(A \otimes B, C) \longrightarrow \mathfrak{K}(A, H(B, C))$ and an isomorphism $b^I : I \otimes I \longrightarrow I$ uniquely determine an SC $(\otimes, H, I, a, b, c, k)$.

5.5. Lemma. Let $\otimes, H$ be given. Then the natural equivalences k^{ABC} are in a one-to-one correspondence with the natural equivalences $\tau : \otimes \longrightarrow \otimes$.

Proof. First, fix a natural equivalence k_0^{ABC} and associate with a general k^{ABC} the natural equivalence $\bar{k} \circ k_0$. Thus, a one-to-one correspondence with the natural equivalence $\mathfrak{K}(A \otimes B, C) \longrightarrow \mathfrak{K}(A \otimes B, C)$ is obtained. Now, for an $e^{ABC} : \mathfrak{K}(A \otimes B, C) \cong \mathfrak{K}(A \otimes B, C)$ define $\tau(e) : \otimes \longrightarrow \otimes$ by $\tau(e)^{AB} = e^{A,B,A\otimes B}(1_{A\otimes B})$. It is easy to check that this is a natural equivalence. On the other hand, for a $t : \otimes \cong \otimes$ define $\varepsilon(t)^{ABC} :$ $: \mathfrak{K}(A \otimes B, C) \longrightarrow \mathfrak{K}(A \otimes B, C)$ putting $\varepsilon(t)^{ABC}(\varphi) =$ $= \varphi \circ t^{AB}$. Again, we see easily that this is a natural equivalence. We have

$$\varepsilon(\tau(e))^{ABC}(\varphi) = \varphi \,.\, e^{A,B,A\otimes B}(1_{A\otimes B}) =$$

$$= (\mathfrak{K}(1, \varphi) \,.\, e^{A,B,A\otimes B})(1) = e^{ABC}(\varphi) \quad ,$$

$$\tau(\varepsilon(t))^{AB} = \varepsilon(t)^{A,B,A\otimes B}(1) = t^{AB} \quad .$$

5.6. Lemma. Let a unit I of a tensor product $\otimes$ be a generator. Then the natural equivalences $\tau : \otimes \longrightarrow \otimes$ are in a one-to-one correspondence with the isomorphisms $\gamma : I \longrightarrow I$.

Proof. Let $\ell^A : A \otimes I \longrightarrow A$ be a natural equivalence. For a natural equivalence $\tau : \otimes \longrightarrow \otimes$ put $\varphi(\tau) = \ell^I \,.\, \tau^{II} \,.\, \bar{\ell}^I$. By 2.5, $\varphi(\tau) = \varphi(\vartheta)$ implies $\tau = \vartheta$. Now, let $\gamma : I \longrightarrow I$ be an arbitrary isomorphism. By 2.4 there is a $\vartheta : 1_{\mathfrak{K}} \cong 1_{\mathfrak{K}}$ with $\vartheta^I = \gamma$.

Put $\tau^{AB} = \vartheta^{A} \otimes 1_B$. We have

$\varphi(\tau) = \ell^{I} . (\gamma \otimes 1_I) . \bar{\ell}^{I} = \gamma$.

5.7. <u>Theorem</u>. Let a tensor structure $(\otimes, \mathcal{H}, I)$ on $\mathfrak{K}$ be given, let I be a generator. Then the SC $(\otimes, \mathcal{H}, I, a, \ell, c, k)$ are in a one-to-one correspondence with the set of couples of isomorphisms $I \longrightarrow I$.

<u>Proof</u>: follows immediately by 5.3, 5.5 and 5.6.

5.8. <u>Corollary</u>. A tensor structure $(\otimes, \mathcal{H}, I)$ on $\mathfrak{K}$ with I a generator without non-identical automorphisms determines uniquely an SC on $\mathfrak{K}$.

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